Closed-form solutions of ECV and ECA tracking filters based on a transfer-function approach

Keigo Watanabe, BEng, MEng, DrEng

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Abstract: New expressions are given for analytical solutions to the steady-state gains of the two-state exponentially correlated velocity (ECV) and three-state exponentially correlated acceleration (ECA) tracking filters with continuous-time position measurements. The present solution method is based on a transfer-function approach, and there is no need to know about the knowledge of the estimation error covariance. For a case when the steady-state error covariance or RMS error value is desired, such gains can be incorporated with the algebraic Riccati equation.

1 Introduction

In relatively simple target tracking problems, it is not always easy to derive closed-form steady-state solutions for the associated Kalman filter gains and the associated error covariance matrix. Such solutions are informative from a theoretical point of view, but are also of practical interest because they allow the numerical computations to be excluded in real time to obtain the gain matrix.

Fitzgerald [1] has already obtained such closed-form steady-state solutions for two particularly simple target tracking filters with continuous-time position measurements, i.e. a two-state exponentially correlated velocity (ECV) and a three-state exponentially correlated acceleration (ECA) filter, respectively. Unfortunately, the approach used by Fitzgerald [1] consists of tedious manipulations of the algebraic Riccati equation and, therefore, sheds no light on the problems under consideration. It was also shown by Gupta and Ahn [2] that this heuristic approach fails to yield a closed-form solution for the discrete-time ECA tracking filter.

In this paper, an alternative solution method based on a transfer-function approach is successfully applied to derive exact closed-form steady-state gains for the continuous-time ECV and ECA tracking filters, without using the knowledge of the estimation error covariance which is represented by the algebraic Riccati equation. However, such gains can be incorporated with the algebraic Riccati equation, to obtain the steady-state error covariance (or RMS error value). The transfer-function method cited here was originally proposed by Grimble [3], and later refined by Flower [4] in continuous-time control problems with single control input.

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The author is with the College of Engineering, Shizuoka University, Johoku 3-5-1, Hamamatsu 432, Japan

2 Review of gain determination via the transfer-function approach

Consider the following linear tracking system with single output:

$$dx(t)/dt = Ax(t) + Bw(t) \tag{1}$$

$$z(t) = Hx(t) + v(t) \tag{2}$$

where $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^p$ and $z(t)$, $v(t) \in \mathbb{R}^1$. The noises $\{w(t),\ v(t)\}$ are zero-mean white Gaussian, but with covariance

$$E\left\{\begin{bmatrix} w(t) \\ v(t) \end{bmatrix}\begin{bmatrix} w^T(\tau) & v(\tau) \end{bmatrix}\right\} = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}\delta(t-\tau) \tag{3}$$

where $Q, R > 0$ and $\delta(.)$ denotes the Dirac delta function. It is well known that the steady-state Kalman filter for eqns. 1–3 becomes [5]:

$$d\hat{x}(t)/dt = A\hat{x}(t) + K[z(t) - H\hat{x}(t)] \tag{4}$$

where the gain K is defined by

$$K \triangleq PH^TR^{-1} \tag{5}$$

where P is the associated estimation error covariance and is a unique positive semidefinite stabilising solution [6] to the following algebraic Riccati equation:

$$AP + PA^T - PH^TR^{-1}HP + BQB^T = 0 \tag{6}$$

In the Riccati equation approach, we must solve eqn. 6 to obtain the optimal gain eqn. 5.

On the other hand, the transfer-function approach [3, 4] directly gives the constant gain K. We briefly outline the algorithm as follows:

(1) Calculate $\phi_0(s) = |sI - A|$

(2) Calculate $G(s) = H\Phi(s)B$ where $\Phi(s) = (sI - A)^{-1}$

(3) Calculate

$$\phi_c(s)\phi_c(-s) = \phi_0(s)\phi_0(-s)[1 + R^{-1}G(s)QG^T(-s)]$$

and factorise the polynomial $\phi_c(s)\phi_c(-s)$ to obtain $\phi_c(s)$, which is a stable polynomial

(4) Compare $\phi_c(s)$ with the equation

$$\phi_c(s) = |sI - A| + \sum_{i=1}^{n} k_i |A_i(H, s)| \tag{7}$$

to obtain k_i, $i = 1, \ldots, n$, where $K = [k_1, \ldots, k_n]^T$ and $|A_i(H, s)|$ represents the determinant $|sI - A|$ with ith row replaced by H.

Notice that this algorithm was accommodated to treat the filtering problem.†

† Watanabe, K.: 'A simple method for solving the constant gains of the Kalman filters with single output', *Int. J. Control*, 1987 (to be published)

3 The ECV case

For this case, the system matrices are given by [7]

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1/\tau \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$H = [1 \quad 0], \quad Q = q_v \qquad (8)$$

where τ denotes the correlation time.

The open-loop characteristic polynomial is given by

$$\phi_0(s) = |sI - A| = \begin{vmatrix} s & -1 \\ 0 & s + 1/\tau \end{vmatrix} = s(s + 1/\tau)$$

and the resolvent matrix is also given by

$$\boldsymbol{\phi}(s) = (sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 0 & s + 1/\tau \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s + 1/\tau)} \\ 0 & \frac{1}{s + 1/\tau} \end{bmatrix}$$

and, therefore, the transfer-function matrix reduces to

$$G(s) = H\boldsymbol{\phi}(s)B = [1 \quad 0] \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s + 1/\tau)} \\ 0 & \frac{1}{s + 1/\tau} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s(s + 1/\tau)}$$

We now have

$$\phi_0(s)\phi_0(-s)[1 + R^{-1}G(s)QG^T(-s)]$$
$$= -s(s + 1/\tau)s(-s + 1/\tau)$$
$$\times \left[1 + R^{-1}q_v \frac{1}{-s(s + 1/\tau)s(-s + 1/\tau)}\right]$$
$$= s^4 - s^2/\tau^2 + q_v/R \qquad \tau, q_v, R > 0$$

Then, it is noted, from the Routh's stability condition, that the stable closed-loop characteristic polynomial $\phi_c(s)$ is of the form

$$\phi_c(s) = s^2 + ms + \sqrt{q_v/R} \qquad m > 0 \qquad (9)$$

whence

$$m^2 = 2\sqrt{q_v/R} + 1/\tau^2$$

or

$$m = \sqrt{2\sqrt{q_v/R} + 1/\tau^2} \qquad (10)$$

On the other hand, we find from eqn. 7 that

$$\phi_c(s) = |sI - A + KH|$$
$$= |sI - A| + \sum_{i=1}^{2} k_i |A_i(H, s)|$$
$$= s(s + 1/\tau) + k_1 \begin{vmatrix} 1 & 0 \\ 0 & s + 1/\tau \end{vmatrix} + k_2 \begin{vmatrix} s & -1 \\ 1 & 0 \end{vmatrix}$$
$$= s^2 + (k_1 + 1/\tau)s + k_1/\tau + k_2 \qquad (11)$$

Finally, comparing this with eqn. 9 yields

$$k_1 = m - 1/\tau \qquad (12)$$

$$k_2 = \sqrt{q_v/R} - m/\tau + 1/\tau^2 \qquad (13)$$

If the estimation error covariance (or RMS error value) is desired, then substituting eqn. 8 into eqns. 5 and 6, and solving P_{11}, P_{12} and P_{22}, we obtain

$$P_{11} = k_1 R \qquad (14)$$

$$P_{12} = k_2 R \qquad (15)$$

$$P_{22} = P_{12}/\tau + P_{11}P_{12}/R \qquad (16)$$

3.1 *Special case (white acceleration model)*

A special case of the above solution is found when τ is allowed to approach infinity. In the two-state case this produces a random walk velocity (RWV) or white acceleration model, for which the steady-state solution is

$$k_1 = m\Big|_{\tau=\infty} = \sqrt{2}(q_v/R)^{1/4} \equiv P_{11}/R \qquad (17)$$

$$k_2 = (q_v/R)^{1/2} \equiv P_{12}/R \qquad (18)$$

$$P_{22}/R = P_{11}P_{12}/R^2 \equiv k_1 k_2 = \sqrt{2}(q_v/R)^{3/4} \qquad (19)$$

This solution entirely coincides with that obtained from the algebraic Riccati equation approach [1]. It is worth noting, however, that the present approach provides a straightforward solution for the ECV case with infinite τ.

4 The ECA case

The corresponding three-state problem involves the ECA model [7], for which the system matrices reduce to

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1/\tau \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$H = [1 \quad 0 \quad 0], \quad Q = q_a \qquad (20)$$

The open-loop characteristic polynomial satisfies the relation

$$\phi_0(s) = |sI - A| = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 0 & s + 1/\tau \end{vmatrix} = s^2(s + 1/\tau)$$

The resolvent matrix $\boldsymbol{\phi}(s)$ is

$$\boldsymbol{\phi}(s) = (sI - A)^{-1} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 0 & s + 1/\tau \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} & \frac{1}{s^2(s + 1/\tau)} \\ 0 & \frac{1}{s} & \frac{1}{s(s + 1/\tau)} \\ 0 & 0 & \frac{1}{s + 1/\tau} \end{bmatrix}$$

so that

$$G(s) = H\boldsymbol{\phi}(s)B$$

$$= [1 \quad 0 \quad 0] \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} & \frac{1}{s^2(s + 1/\tau)} \\ 0 & \frac{1}{s} & \frac{1}{s(s + 1/\tau)} \\ 0 & 0 & \frac{1}{s + 1/\tau} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^2(s + 1/\tau)}$$

Therefore, we can obtain

$$\phi_0(s)\phi_0(-s)[1 + R^{-1}G(s)QG^T(-s)]$$
$$= s^2(s + 1/\tau)s^2(-s + 1/\tau)$$
$$\times \left[1 + R^{-1}q_a \frac{1}{s^2(s+1/\tau)s^2(-s+1/\tau)}\right]$$
$$= -s^6 + (1/\tau^2)s^4 + q_a/R \qquad \tau, q_a, R > 0$$

It is further clear, from the Routh's stability condition, that the stable $\phi_c(s)$ must be of the form

$$\phi_c(s) = s^3 + m_1 s^2 + m_2 s + \sqrt{q_a/R}$$
$$m_1, m_2 > 0 \quad \text{and} \quad m_1 m_2 - \sqrt{q_a/R} > 0 \tag{21}$$

whence

$$m_1 = \sqrt{1/\tau^2 + 2m_2} > 0 \tag{22}$$

and m_2 is one positive real root for the quartic equation

$$m_2^4 - 8\frac{q_a}{R} m_2 - \frac{4q_a}{R\tau^2} = 0 \tag{23}$$

Such a solution can then be determined, using the procedure given in the Appendix, as

$$m_2 = \tfrac{1}{2}[\sqrt{\lambda_1} + \sqrt{\lambda_1 - 2(\lambda_1 - \sqrt{\lambda_1^2 + 16q_a/(R\tau^2)})}] \tag{24}$$

where λ_1 is one of the real roots of

$$\lambda^3 + \frac{16q_a}{R\tau^2}\lambda - \frac{64q_a^2}{R^2} = 0 \tag{25}$$

The cubic polynomial of eqn. 25 has one real root

$$\lambda_1 = U + V \tag{26}$$
$$U = \sqrt[3]{32q_a^2/R^2 + \sqrt{D}} \tag{27}$$
$$V = \sqrt[3]{32q_a^2/R^2 - \sqrt{D}} \quad * \tag{28}$$

because

$$D = \left(\frac{16q_a}{3R\tau^2}\right)^3 + \left(\frac{32q_a^2}{R^2}\right)^2 > 0 \tag{29}$$

On the other hand, substituting eqn. 20 into eqn. 7 and performing the determinants yields

$$\phi_c(s) = |sI - A + KH|$$
$$= |sI - A| + \sum_{i=1}^{3} k_i |A_i(H, s)|$$
$$= s^2(s + 1/\tau) + k_1 \begin{vmatrix} 1 & 0 & 0 \\ 0 & s & -1 \\ 0 & 0 & s + 1/\tau \end{vmatrix}$$
$$+ k_2 \begin{vmatrix} s & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & s + 1/\tau \end{vmatrix} + k_3 \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 0 & 0 \end{vmatrix}$$
$$= s^3 + (1/\tau + k_1)s^2$$
$$+ (k_1/\tau + k_2)s + k_2/\tau + k_3 \tag{30}$$

Comparing this with eqn. 21, it is simple to calculate k_1, k_2 and k_3, namely

$$k_1 = m_1 - 1/\tau \tag{31}$$
$$k_2 = m_2 - m_1/\tau + 1/\tau^2 \tag{32}$$
$$k_3 = \sqrt{q_a/R} - m_2/\tau + m_1/\tau^2 - 1/\tau^3 \tag{33}$$

If we use eqns. 5, 6 and 20, then the estimation error covariance can also be reconstructed in terms of k_1, k_2 and k_3 as

$$P_{11} = k_1 R \tag{34}$$
$$P_{12} = k_2 R \tag{35}$$
$$P_{13} = k_3 R \tag{36}$$
$$P_{22} = P_{11}P_{12}/R - P_{13} \tag{37}$$
$$P_{23} = P_{13}/\tau + P_{11}P_{13}/R \tag{38}$$
$$P_{33} = P_{23}/\tau + P_{12}P_{13}/R \tag{39}$$

Notice that the above solutions are entirely different in the expression from those of Fitzgerald [1], but both solutions must have the same values for the specified parameters τ, q_a and R.

4.1 *Special case (white jerk model)*

In the three-state ECA case with infinite τ, we obtain a random walk acceleration (RWA) or a white jerk model. The solution approaches, in the limit,

$$k_2 = m_2\Big|_{\tau=\infty} = \sqrt{\lambda_1}\Big|_{\tau=\infty} = 2(q_a/R)^{1/3} \equiv P_{12}/R \tag{40}$$
$$k_1 = m_1\Big|_{\tau=\infty} = \sqrt{2m_2}\Big|_{\tau=\infty} = 2(q_a/R)^{1/6} \equiv P_{11}/R \tag{41}$$
$$k_3 = (q_a/R)^{1/2} \equiv P_{13}/R \tag{42}$$
$$P_{22}/R = P_{11}P_{12}/R^2 - P_{13}/R \equiv k_1k_2 - k_3$$
$$= 3(q_a/R)^{1/2} \tag{43}$$
$$P_{23}/R = P_{11}P_{13}/R^2 \equiv k_1k_3 = 2(q_a/R)^{2/3} \tag{44}$$
$$P_{33}/R = P_{12}P_{13}/R^2 \equiv k_2k_3 = 2(q_a/R)^{5/6} \tag{45}$$

Once again, it should be noted that the above solutions can be derived by simple inspection of eqns. 34–39 with $\tau = \infty$, and are completely identical to the solutions presented by Reference 1.

It is of interest to compare the characteristics of two analytical solutions. Tables 1 and 2 show the typical gains as functions of q_a/R and τ, where they were generated by using a program written by Fortran 77 language in single precision arithmetic. From the Tables, we perceive that the analytical solution due to the Riccati equation method gives erroneous values for large values of τ,

Table 1: Gains for the ECA tracking filter: Riccati equation method [1], where $q_a/R = 1$

τ	k_1	k_2	k_3
10^{-2}	1.414214×10^{-1}	1.000000×10^{-2}	4.992939×10^{-7}
0.1	4.471116×10^{-1}	9.995441×10^{-2}	4.781649×10^{-4}
1	1.299870	8.448309×10^{-1}	1.551695×10^{-1}
10	1.903334	1.811341	8.188738×10^{-1}
10^2	1.990033	1.980116	9.801987×10^{-1}
10^3	*	*	*
10^4	*	*	*

Notice that asterisks denote erroneous or unsatisfactory values.

Table 2: Gains for the ECA tracking filter: transfer-function method, where $q_a/R = 1$

τ	k_1	k_2	k_3
10^{-2}	1.414162×10^{-1}	1.021266×10^{-1}	*
0.1	4.471075×10^{-1}	9.994209×10^{-2}	5.329847×10^{-4}
1	1.299870	8.448305×10^{-1}	1.551695×10^{-1}
10	1.904981	1.814477	8.185523×10^{-1}
10^2	1.990050	1.980150	9.801985×10^{-1}
10^3	1.999000	1.998001	9.980021×10^{-1}
10^4	1.999900	1.999800	9.998000×10^{-1}

while the present solution provides an unsatisfactory value for small values of τ. These are caused by the numerically rounded errors. It is found, however, from Figs. 1 to 3 that both analytical solutions produce exactly

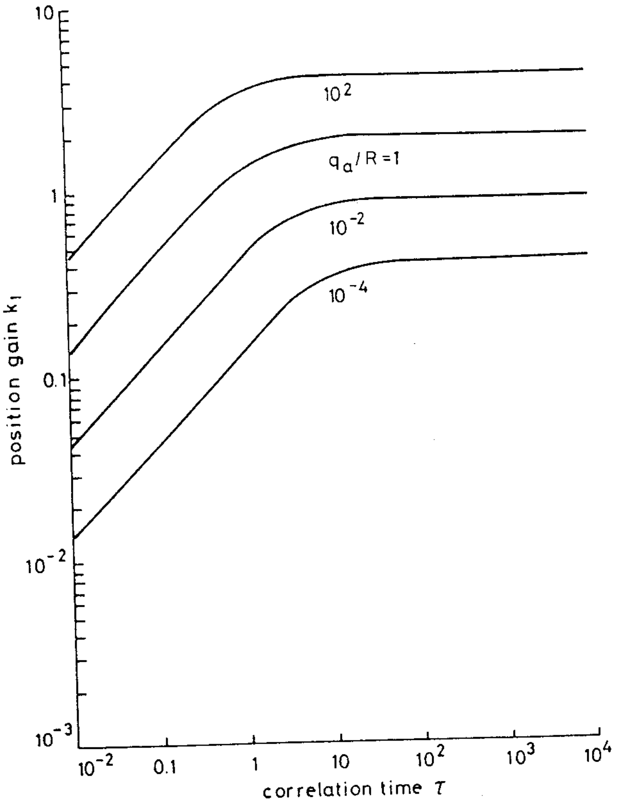

Fig. 1 *Position gain for the ECA tracking filter*

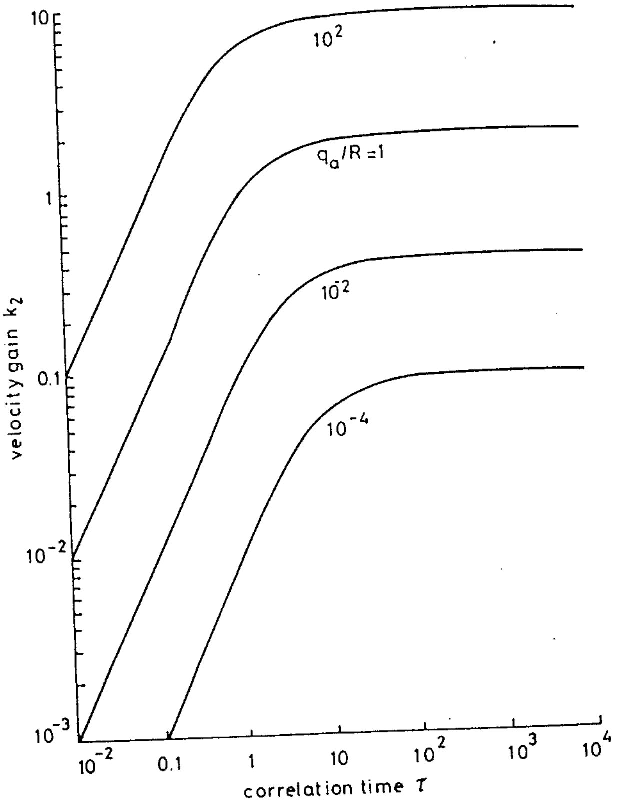

Fig. 2 *Velocity gain for the ECA tracking filter*

the same gain values in double precision arithmetic. Additionally, we can graphically confirm that the solutions 31–33 converge to the limiting values 40–42 when the correlation time τ approaches infinity.

5 Conclusions

Steady-state gains have been derived for continuous-time ECV and ECA tracking filters without using any knowledge of the estimation error covariance. The present

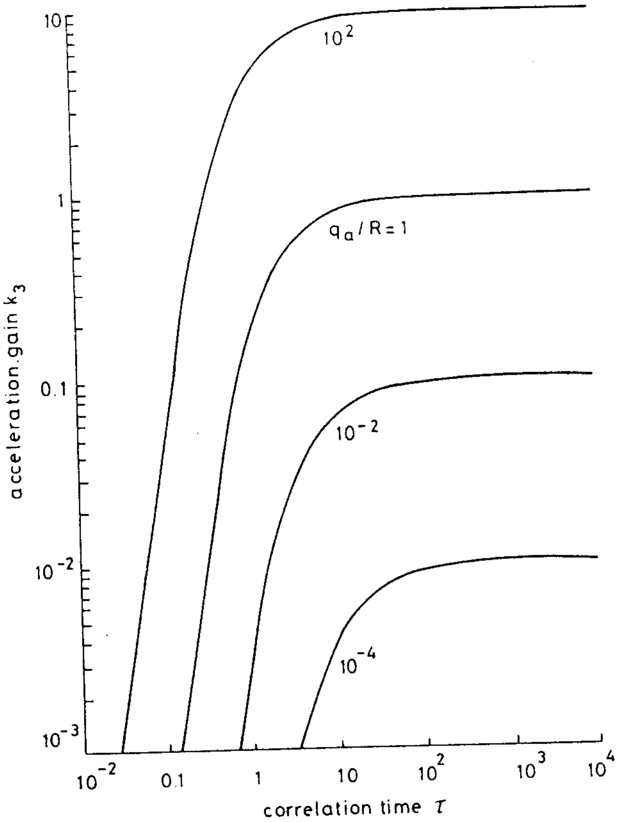

Fig. 3 *Acceleration gain for the ECA tracking filter*

approach based on the transfer-function approach gives answers to both ECV and ECA problems in a systematic manner. Also, it has been shown that closed-form steady-state solutions for the estimation error covariance can be constructed in terms of such gains.

When the present approach is directly extended to discrete-time ECV and ECA problems as studied by Gupta and Ahn [2], there may be difficulty in obtaining equations similar to eqns. 9 and 21, because of less sparsity of the transition matrices. However, an approach which combines the present method with the technique of bilinear transformation [8] may present no particular difficulty.

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7 Appendix

The objective of this appendix is to give the solutions of quartic and cubic equations [9, 10].

If $x^4 + ax^3 + bx^2 + cx + d = 0$, then the roots are

$$x_{1,2} = [p_1 \pm \sqrt{p_1^2 - 4q_1}]/2$$
$$x_{3,4} = [p_2 \pm \sqrt{p_2^2 - 4q_2}]/2$$

where

$$p_{1,2} = \pm\sqrt{a^2/4 + m - b} - a/2$$
$$q_{1,2} = (m \mp \sqrt{m^2 - 4d})/2$$

and m is one of the real roots of

$$m^3 - bm^2 + (ac - 4d)m + (4bd - c^2 - a^2 d) = 0$$

Let $M = m - b/3$. Then the cubic equation is transformed to

$$M^3 + pM + q = 0$$

where

$$p = (ac - 4d - b^2/3)$$
$$q = (4bd - c^2 - a^2 d) + (b/3)(ac - 4d) - 2b^3/27$$

The roots of the M equation are

$$M_1 = u + v$$
$$M_2 = -(u + v)/2 + (u - v)i\sqrt{3}/2$$
$$M_3 = -(u + v)/2 - (u - v)i\sqrt{3}/2$$

where

$$u = \sqrt[3]{-q/2 + \sqrt{D}}$$
$$v = \sqrt[3]{-q/2 - \sqrt{D}}$$
$$D = (p/3)^3 + (q/2)^2$$

(i) $D > 0$. One root M_1 is real and other two are complex conjugate.

(ii) $D = 0$. There are three real roots; at least two are equal:

$$M_1 = 2\sqrt[3]{-q/2}$$
$$M_2 = M_3 = -\sqrt[3]{-q/2}$$

(iii) $D < 0$. There are three real and unequal roots:

$$M_1 = 2\sqrt[3]{\rho}\cos(\phi/3)$$
$$M_2 = 2\sqrt[3]{\rho}\cos(\phi/3 + 2\pi/3)$$
$$M_3 = 2\sqrt[3]{\rho}\cos(\phi/3 + 4\pi/3)$$

where

$$\rho = \sqrt{-p^3/27}, \quad \cos\phi = -q/2p$$

$$* \sqrt{D} \geq \frac{32 q_a^2}{R^2}$$

so rather use

$$V = -\sqrt[3]{\sqrt{D} - \frac{32 q_a^2}{R^2}}$$