Uncertainty

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1 Uncertainty

1.1 Uncertainty Descriptions

Three descriptions of uncertainty are commonly used:

$$\begin{array}{rrl} \text{Additative} & G(s) = & G_0(s) + \Delta_a(s) \\ \text{Input Multiplicative} & G(s) = & G_0(s)[I + \Delta_i(s)] \\ \text{Output Multiplicative} & G(s) = & [I + \Delta_o(s)]G_0(s) \end{array}$$

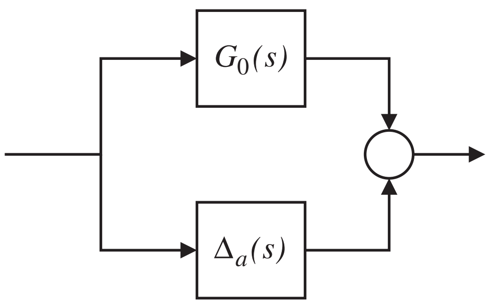
The additive model means that for $||\Delta_a||_\infty = 0.1$ that

$$||G - G_0||_\infty \leq 0.1$$

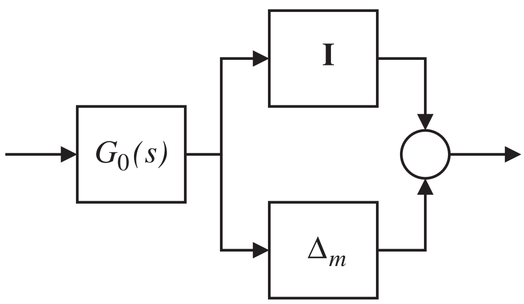
and the multiplicative model with $||\Delta_m||_\infty = 0.1$ means that

$$||G - G_0||_\infty = ||G_0\Delta_m||_\infty \leq ||G_0||_\infty \, ||\Delta_m||_\infty \leq 0.1||G_0||_\infty$$

In block diagram format, the additive and output multiplicative uncertainty can be drawn as follows.



Additive



Multiplicative

The textbooks uses a generic form for Δ_P which makes it easier to describe uncertainties in terms of **unit** uncertainty Δ, and scaling and a structure of the effect of the uncertainty components through input and output weights $W_1,\ W_2$ by using

$$\Delta_P = W_1 \Delta W_2, \quad \overline{\sigma}[\Delta(j\omega)] = ||\Delta(j\omega)||_\infty \leq 1,\ \forall\omega \geq 0$$

Example Consider the following uncertain plant

$$P(s) = \frac{a}{s+b}, \quad 5 \leq a \leq 7,\ 2 \leq b \leq 4$$

The four extreme transfer functions would be

$$P_1(s) = \frac{5}{s+2}, \quad P_2(s) = \frac{7}{s+2}, \quad P_3(s) = \frac{5}{s+4}, \quad P_4(s) = \frac{7}{s+4},$$

Choose for example a nominal plant with transfer function

$$P_0(s) = \frac{6}{s+3}$$

then the additive uncertainty is given by

$$\Delta_a = P(s) - P_0(s) = \frac{a}{s+b} - \frac{6}{s+3} = \frac{(a-6)s + 3a - 6b}{(s+b)(s+3)}$$

By inspection, the maximum transfer function occurs at the minimum value of a and maximum value of b and decreases with frequency (more poles than zeroes) with

$$||\Delta_a||_\infty < 1.7143$$

Similarly, the multiplicative uncertainty is given by

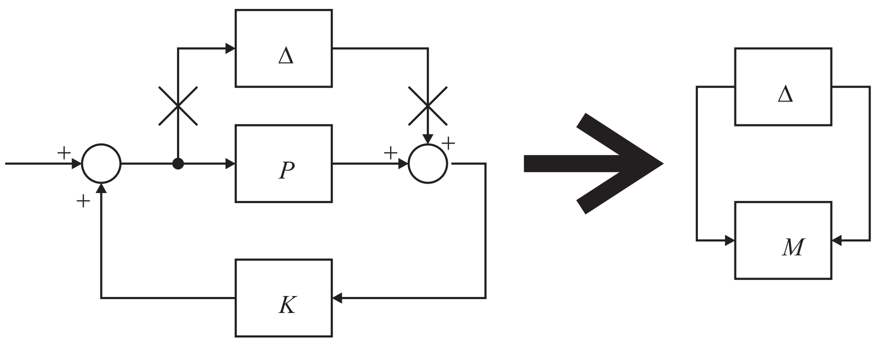
$$\Delta_m = \frac{P(s) - P_0(s)}{P_0(s)} = \frac{\frac{a}{s+b} - \frac{6}{s+3}}{\frac{6}{s+3}} = \frac{(a-6)s + 3a - 6b}{6(s+b)}$$

By inspection

$$||\Delta_m||_\infty < 0.8571$$

1.2 Uncertainty and Stability

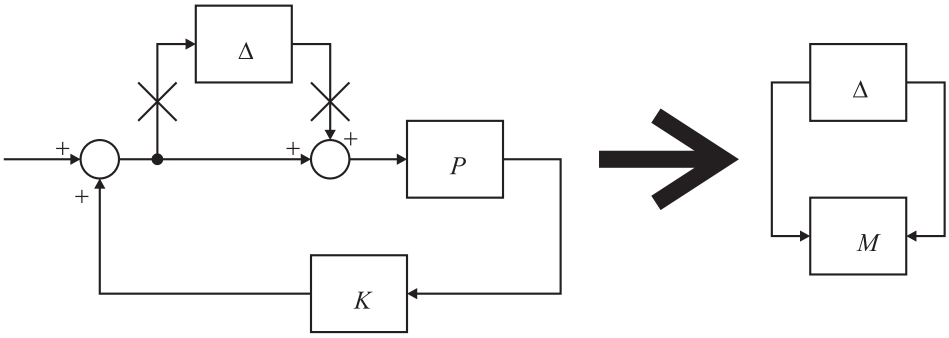
Consider the following control loop with additive uncertainty



For this control system, the “gain” seen by the uncertainty is now

$$M = K(I - PK)^{-1}$$

A similar situation can be drawn for the multiplicative uncertainty



For this control system, the “gain” seen by the uncertainty is now

$$M = KP(I - KP)^{-1}$$

This process is called extracting the uncertainty or pulling out the “Δ(s)”.

To analyze the the stability of the control system, we can now use the **small gain theorem** which states that if $\Delta \in \mathcal{RH}_\infty$ and $M \in \mathcal{RH}_\infty$ (i.e. both are stable and bounded), then the interconnected system is stable if

$$||\Delta M||_\infty < 1 \quad \text{or} \quad ||\Delta||_\infty ||M||_\infty < 1$$

The boundary operator γ is used here to indicate how the bounds of M and Δ are limited, i.e.

$$||\Delta||_\infty \leq \frac{1}{\gamma} \quad \text{if} \quad ||M||_\infty < \gamma$$

1.3 Unstructured uncertainty

The type of uncertainty we are interested here is one where the gain or transfer function is bounded between certain limits, but the origin of the variations are not described.

For this type of uncertainty, the additive and multiplicative uncertainty models are used.

The maximum value of the uncertainty is contained by $||\Delta||_\infty$, or if we know the frequency dependence by $\overline{\sigma}[\Delta(s)]$.

By using two weighting matrices W_1 and W_2, we can overcome the non-commutative nature of matrix multiplication to represent simultaneous input and output multiplicative uncertainty specifications

$$||W_1^{-1} \Delta W_2^{-1}||_\infty \leq 1$$

Table 8.1 in the textbook has a very useful compilation of the effect of the different uncertainties and the necessary uncertainty tests for robust stability. See also example 8.3 below how to describe uncertainty for simple systems.

Example 8.3 Consider the following uncertain plant

$$P_\Delta(s) = \frac{1}{s-\delta}, \quad |\delta| \leq 1$$

Through manipulation we have with $P(s) = 1/s$

$$P_\Delta = \frac{1}{s-\delta} = \frac{1}{s(1-\delta/s)} = P(s)\,\frac{1}{1-P(s)\,\delta} = P(s)\,(1-P(s)\,\delta)^{-1}$$

This uncertainty falls under the category of $P(I+W_1\Delta W_2)^{-1}$ in Table 8.1, row 4 and the robust stability test is therefore

$$||W_2 S_i W_1||_\infty \leq 1$$

1.4 Structured uncertainty

For this type of uncertainty the structure of the changes are well known e.g. the uncertainty in input 1 has a bound and does not influence the other inputs.

A concern for structured uncertainties is that if our bound is too large, we lose the structure information.

Example

$$\Delta_i = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}, \quad |\delta_i| \leq 0.1$$

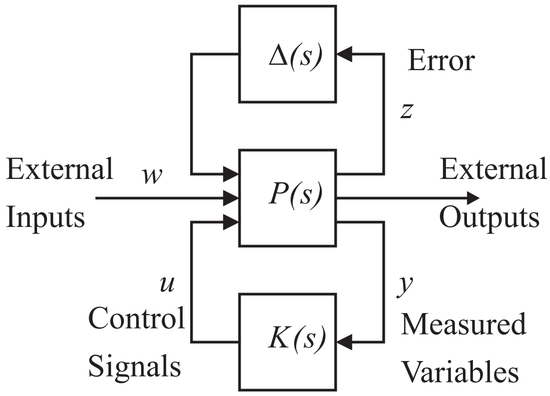
This structured uncertainty is very specific, input 1 drives only output 1 etc. If our weighing matrix W is too loose, say $||10\Delta_i||_\infty \leq 1$, the solution might be satisfied by pertubations such as

$$\Delta_i = \begin{bmatrix} 0 & 0.1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \Delta_i = \frac{1}{\sqrt{2}} \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix}$$

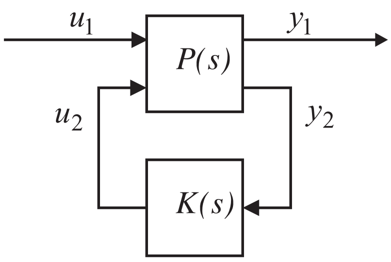
2 Linear Fractional Transformation

Although we can describe every system from initial description and break it down into a uncertain transfer function Δ and a certain system transfer function M, it would be nice if we can derive a general transformation that works in general. The linear fractional transformation is such a description.

Consider the following full system diagram



which includes a nominal plant $P(s)$, plant uncertainty $\Delta(s)$ and a controller $K(s)$. Consider for a moment only the nominal plant and controller



with The plant

$$P(s) = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

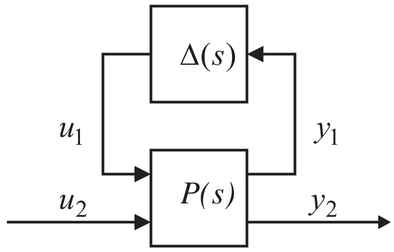
The equations for the lower system is now

$$\begin{aligned} y_1 &= P_{11}u_1 + P_{12}u_2 \\ y_2 &= P_{21}u_1 + P_{22}u_2 \\ u_1 &= Ky_2 \end{aligned}$$

We can solve the transfer function from u_1 to y_1 to give the lower linear fractional transformation (LFT)

$$\begin{aligned} y_1 &= \left[P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}\right] u_1 \\ &\equiv F_l(P, K)u_1 \end{aligned}$$

Similarly, we can get the upper linear fractional transformation of P and Δ by



$$\begin{aligned} y_1 &= P_{11}u_1 + P_{12}u_2 \\ y_2 &= P_{21}u_1 + P_{22}u_2 \\ u_1 &= \Delta y_1 \end{aligned}$$

which gives

$$\begin{aligned} y_2 &= \left[P_{22} + P_{21}\Delta(I - P_{11}\Delta)^{-1}P_{12}\right] u_2 \\ &\equiv F_u(P,\Delta)u_2 \end{aligned}$$

Consider the following state-space description

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right], \quad K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right], \quad \Delta = \left[\begin{array}{c|c} A_\Delta & B_\Delta \\ \hline C_\Delta & D_\Delta \end{array}\right]$$

The lower LFT is then given by

$$F_l(P,K) = \left[\begin{array}{cc|c} A + B_2FD_KC_2 & B_2FC_K & B_1 + B_2FD_KD_{21} \\ B_KEC_2 & A_K + B_KED_{22}C_K & B_KED_{21} \\ \hline C_1 + D_{12}FD_KC_2 & D_{12}FC_K & D_{11} + D_{12}FD_KD_{21} \end{array}\right]$$

with $E = (I - D_{22}D_K)^{-1}$ and $F = (I - D_KD_{22})^{-1}$.

Similarly the upper LFT is given by

$$F_u(P,\Delta) = \left[\begin{array}{cc|c} A + B_1FD_\Delta C_1 & B_1FC_\Delta & B_2 + B_1FD_\Delta D_{12} \\ B_\Delta EC_1 & A_\Delta + B_\Delta ED_{11}C_\Delta & B_\Delta ED_{12} \\ \hline C_2 + D_{21}FD_\Delta C_1 & D_{21}FC_\Delta & D_{22} + D_{12}FD_\Delta D_{12} \end{array}\right]$$

with $E = (I - D_{11}D_\Delta)^{-1}$ and $F = (I - D_\Delta D_{11})^{-1}$.

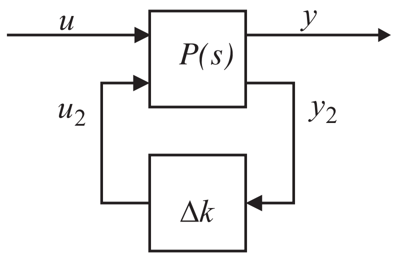
Example Consider a mass-spring model

$$m\ddot{y} = -ky + f$$

The transfer function is now

$$G(s) = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -k/m & 0 & 1/m \\ \hline 1 & 0 & 0 \end{array}\right]$$

Now say the spring constant changes by an amount Δk from the nominal value k_n, then we can write $k - k_n + \Delta k$ (additive model). Our second row then becomes $\dot{x}_2 = -\frac{k_n}{m}x_1 + \frac{1}{m}u - \frac{\Delta k}{m}x_1$. To write it in a lower LFT format



create a new fictitious output $y_2 = x_1$ and fictitious input u_2 that drives the $\dot{x}_2$ equation. The resulting $P(s)$ becomes

$$P(s) = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ \frac{-k_n}{m} & 0 & \frac{1}{m} & \frac{-1}{m} \\ \hline 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right]$$

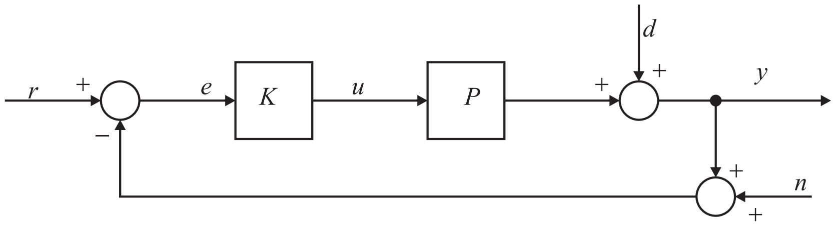
The transfer function from the lower LFT can be verified to give again $G(s)$

$$G(s) = F_l(P, \Delta k) = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ \frac{-(k_n+\Delta k)}{m} & 0 & \frac{1}{m} \\ \hline 1 & 0 & 0 \end{array}\right]$$

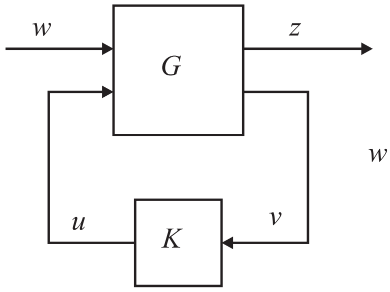
In general how do we do this? The following process is proposed:

1. Draw a block diagram of the system with the δ elements separated (additive or multiplicative).
2. Mark the inputs and outputs of the δ elements as y's and u's respectively.
3. Write the rest as z's and y's in terms of w and u's with all δ's taken out.

Example Consider the following block diagram

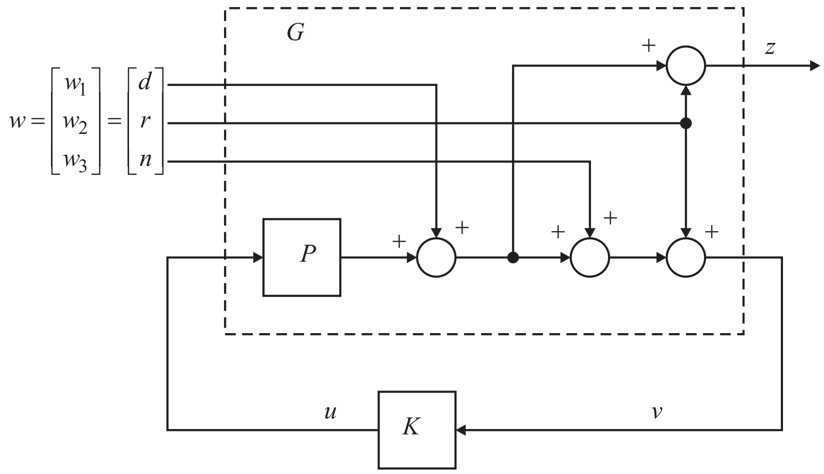


We want to get the system in the following LFT structure



$$w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} d \\ r \\ n \end{bmatrix}, \quad z = e = y - r, \quad v = r - y - n$$

By re-drawing this structure we get the diagram



G does not contain the controller and is the transfer function from $[w\ u]^T$ to $[z\ v]^T$ and is given by

$$G = \begin{bmatrix} I & -I & 0 & P \\ -I & I & -I & -P \end{bmatrix}$$

To enter this system using low-level commands in Matlab to `sysic`, we can write

```
systemnames = 'P';
inputvar    = '[d(1);r(1);n(10u(10]';
input_to_P  = '[u]';
```

```
outputvar   = '[P+d-r; r-P-d-n]';
sysoutname  = 'G';
sysic;
```

Homework Problems 8.1 and 9.3.

8.1 This problem illustrates that the stability margin is dependent on the type of pertubation. System is unity feedback with $K(s) = 1$ and $P(s) = P_{\text{nom}} + \Delta(s)$ where

$$P_{\text{nom}} = \frac{10}{s^2 + 0.2\,s + 1}$$

1. Assume $\Delta(s) \in RH_\infty$ (i.e. no poles in RH plane). Compute the largest β such that the feedback is internally stable for all $||\Delta||_\infty < \beta$.
2. Repeat for all $\Delta(s) \in R$.

9.3 Consider the unity feedback system with $G(s)$ with uncertainty model

$$G(s) = \begin{bmatrix} (1 + \Delta_{11}(s))g_{11}(s) & (1 + \Delta_{12}(s))g_{12}(s) \\ (1 + \Delta_{21}(s))g_{21}(s) & (1 + \Delta_{22}(s))g_{22}(s) \end{bmatrix}$$

Pull out the delta's and draw the diagram.