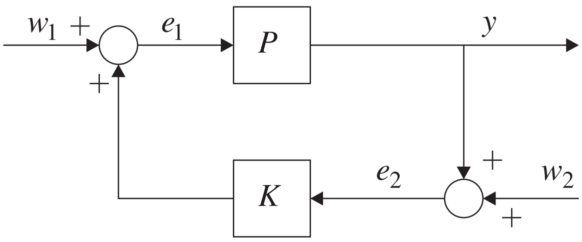
Robust Stability

J Treurnicht

May 8, 2007

1 Internal Stability

Consider the following system (using a positive feedback notation)



From the figure

$$\begin{aligned} e_1 &= w_1 + Ke_2 \\ e_2 &= w_2 + Pe_1 \end{aligned}$$

and solving for e_1 we get

$$e_1 = w_1 + K(w_2 + Pe_1), \quad \rightarrow \quad [I - KP]e_1 = w_1 + Kw_2$$

so writing it as a matrix

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} I & K \\ P & I \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

From algebra

$$\begin{aligned}
e_1 = w_1 + K(w_2 + Pe_1) \quad &\rightarrow \quad \frac{e_1}{w_1}(s) = H_{11}(s) = [I - K(s)P(s)]^{-1} \\
&\rightarrow \quad \frac{e_1}{w_2}(s) = H_{12}(s) = K(s)[I - K(s)P(s)]^{-1} \\
e_2 = w_2 + P(w_1 + Ke_2) \quad &\rightarrow \quad \frac{e_2}{w_1}(s) = H_{21}(s) = P(s)[I - P(s)K(s)]^{-1} \\
&\rightarrow \quad \frac{e_2}{w_2}(s) = H_{22}(s) = [I - P(s)K(s)]^{-1}
\end{aligned}$$

As an example consider the system with $P(s) = \frac{-s}{s+1}$ and $K(s) = \frac{s+3}{s(s+4)}$. The return ratio

$$H_{11}(s) = [1 - KP]^{-1} = \left[1 + \frac{s+3}{(s+1)(s+4)}\right]^{-1} = \left[\frac{s^2+6s+7}{(s+1)(s+4)}\right]^{-1} = \frac{(s+1)(s+4)}{s^2+6s+7}$$

is stable but the transfer function to output not

$$\frac{y_1}{w_2}(s) = K\,(1 - PK)^{-1} = \frac{s+3}{s(s+4)} \cdot \frac{(s+1)(s+4)}{s^2+6s+7} = \frac{(s+3)(s+1)}{s(s^2+6s+7)}$$

which has a pole at the origin (on imaginary axis) and is not strictly stable.

Definition 1 *The feedback system in the figure is internally stable if and only if the transfer matrix*

$$\begin{bmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \end{bmatrix}$$

is exponentially stable in the equation

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

(the transfer functions from every (w_1, w_2) to every (e_1, e_2) are stable)

The purpose of this definition is to exclude right-half plane pole-zero cancellations between $P(s)$ and $K(s)$ — which cannot be detected with Nyquist tests that operate on the return ratio $GH(s)$. This implies that all four elements H_{ij} must be exponentially stable.

If both $P(s)$ and $K(s)$ are unstable, we will have to calculate all four elements to determine stability.

If one element is stable such as $K(s)$, then there is less work to do. If in this case $H_{21}(s) = P(s)[I - PK(s)]^{-1}$ is stable then the total system is stable.

Assume $K(s)$ and $H_{21}(s)$ is stable. Then H_{22} is also stable (same pole positions).

As the transfer functions $P(s)$ and $K(s)$ is scalar, $P(s)K(s) = K(s)P(s) = PK(s)$ and the poles of the transfer functions are the same. The stability of $H_{21}(s)$ is determined by the poles of the composite transfer function $[I - P(s)K(s)]^{-1}$. If $H_{21}(s)$ and $K(s)$ is stable, then H_{ij} is also stable.

The following Nyquist-like criteria applies to this system: If $K(s)$ is exponentially stable, then $H_{21}(s)$ is exponentially stable if and only if

- det $[I - P(s)K(s)]^{-1}$ has no zeroes in the closed right-half plane (CRHP)(including infinity)
- $P(s)[I - P(s)K(s)]^{-1}$ is analytic (no poles) at every closed right-half pole of $G(s)$ (including infinity)

The same is done in the textbook (Zhou) by looking at the stability of

$$\begin{bmatrix} I & -K \\ -P & I \end{bmatrix}^{-1} = \begin{bmatrix} (I - KP)^{-1} & K(I - KP)^{-1} \\ P(I - KP)^{-1} & (I - KP)^{-1} \end{bmatrix}$$

This will ensure that all outputs are bounded for all inputs.

Example 5.1 With P and K as two-by-two transfer matrices

$$P(s) = \begin{bmatrix} \frac{1}{s-1} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix}, \quad K(s) = \begin{bmatrix} \frac{1-s}{s+1} & -1 \\ 0 & -1 \end{bmatrix}$$

we get

$$PK(s) = \begin{bmatrix} \frac{-1}{s+1} & \frac{-1}{s-1} \\ 0 & \frac{-1}{s+1} \end{bmatrix}, \quad (I - PK(s))^{-1} = \begin{bmatrix} \frac{s+1}{s+2} & -\frac{(s+1)^2}{(s+2)^2(s-1)} \\ 0 & \frac{s+1}{s+2} \end{bmatrix}$$

Even if $\det(I - PK) = \frac{(s+1)^2}{(s+2)^2}$ has no zeros in the RHP and the number of unstable poles of $PK = 1$, this system is not internally stable.

2 Generalized Nyquist Stability Criteria

If $G(s)$ has n unstable (Smith-McMillan) poles, then the closed loop system with return ratio $-KG(s)$ is stable if and only if the characteristic loci of

$KG(s)$, taken together, encircle the $-1 + j0$ point n times anti-clockwise, assuming that there are no hidden unstable poles.

Example

$$G(s) = \begin{bmatrix} 0 & 1 \\ \frac{s-1}{s+1} & 0 \end{bmatrix}$$

This function has eigenvalues at $\lambda_1 = \pm\sqrt{(s-1))/(s+1)}$.

3 Coprime Factorization

Two transfer functions $m(s)$ and $n(s)$ are coprime if we can find a x, y such that

$$x(s)m(s) + y(s)n(s) = 1$$

Depending on the order xm or mx, we get right and left coprime matrices.

Consider the plant

$$P(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

A right coprime factorization of $P(s)$ is

$$P(s) = N_r(s)M_r^{-1}(s)$$

where N_r and M_r are stable coprime transfer functions. There must be no common factors in N_r and M_r that may cancel RHP poles.

A left coprime factorization of $P(s)$ is

$$P(s) = M_l^{-1}(s)N_l(s)$$

where N_l and M_l are stable coprime transfer functions.

Example Consider the scalar function

$$P(s) = \frac{(s-1)(s+2)}{(s-3)(s+4)}$$

To obtain coprime factorization, make all the RHP poles of P zeroes of M and all the RHP zeroes of P zeroes of N. Then we allocate the poles of N and M so that they are proper and the identity $G = NM^{-1}$ holds

$$N(s) = \frac{s-1}{s+4}, \quad M(s) = \frac{s-3}{s+2}$$

Usually we select N and M to have the same poles

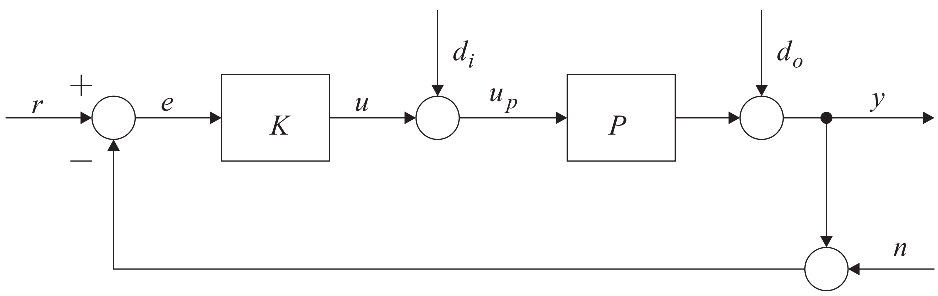
$$N(s) = \frac{(s-1)(s+2)}{s^2 + k_1 s + k_2}, \quad M(s) = \frac{(s-3)(s+4)}{s^2 + k_1 s + k_2}, \quad k_1, k_2 > 0$$

Note that coprime factorization is not unique.

4 Performance Specifications

4.1 Loop Parameters

Consider the following control system with measurement noise n, and disturbance inputs d_i and d_o.



If we define the input loop transfer matrix $L_i = KP$ (breaking the loop at the input u) and the output loop transfer function $L_o = PK$ (breaking the loop at the output y), then we can write the sensitivities as

$$\begin{aligned} S_i &= (I + L_i)^{-1} = (I + KP)^{-1} & u &= S_i d_i \\ S_o &= (I + L_o)^{-1} = (I + PK)^{-1} & y &= S_o d_o \\ T_i &= I - S_i = L_i (I + L_i)^{-1} \\ T_o &= I - S_o = L_o (I + L_o)^{-1} \end{aligned}$$

The transfer functions becomes

$$\begin{aligned} y &= T_o(r - n) + S_o P d_i + S_o d_o \\ e &= S_o(r - d) + T_o n - S_o P d_i \\ u &= K S_o (r - n) - K S_o d_o - T_i d_i \end{aligned}$$

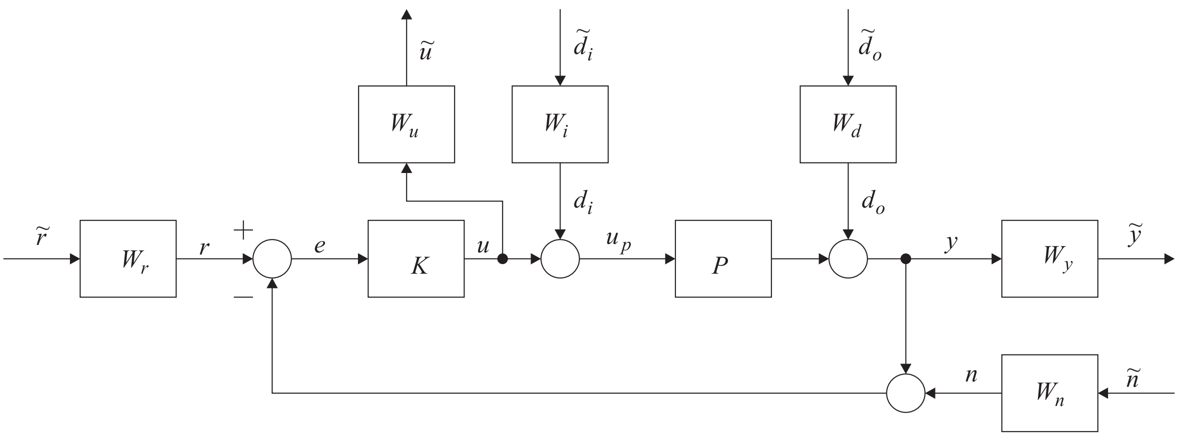
For good disturbance rejection on the output y, we will require that S_o be small and T_o be both large (r) and small (n). We usually split this problem into two frequency bands and end up with something along the line

$$\begin{array}{rcccl} S_o < & A & \forall & \omega \leq & \omega_0 \\ T_o > & B & \forall & \omega \leq & \omega_0 \\ T_o < & C & \forall & \omega \geq & \omega_1 \end{array}$$

You can also express it in terms of singular values as shown on pages 82 and 83.

4.2 Weights

We can formalize the process of specification by defining weights as shown below for our control loop



Let us specify the sensitivity function as follows

$$S(j\omega) \left\{ \begin{array}{ll} |S(j\omega)| \leq \epsilon, & \forall \omega \leq \omega_0 \\ |S(j\omega)| \leq M, & \forall \omega \geq \omega_1 \end{array} \right.$$

We can rather choose a weighting function W_y so that

$$|W_y(j\omega)S(j\omega)| \leq 1, \ \forall \omega$$

with

$$|W_y(j\omega)| = \left\{ \begin{array}{ll} \frac{1}{\epsilon}, & \forall \omega \leq \omega_0 \\ \frac{1}{M}, & \forall \omega \geq \omega_1 \end{array} \right.$$

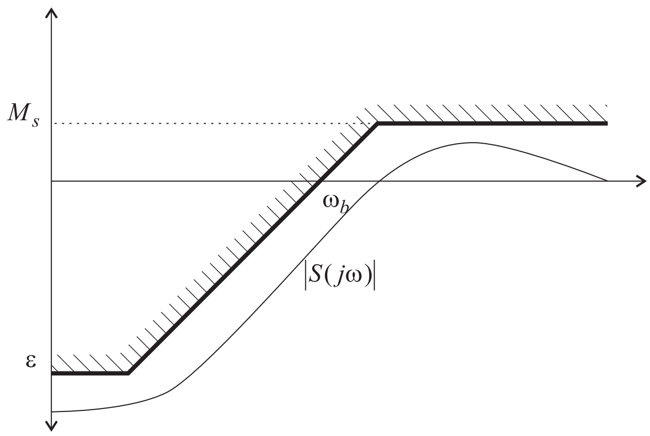
For the selection of practical weighting functions, can go back to our second order prototypes

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad \rightarrow \quad L = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}, \quad S = \frac{s(s + 2\zeta\omega_n)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

A good simple practical weighting function for the sensitivity S is

$$W_y = \frac{s/M_s + \omega_b}{s + \omega_b \epsilon}$$

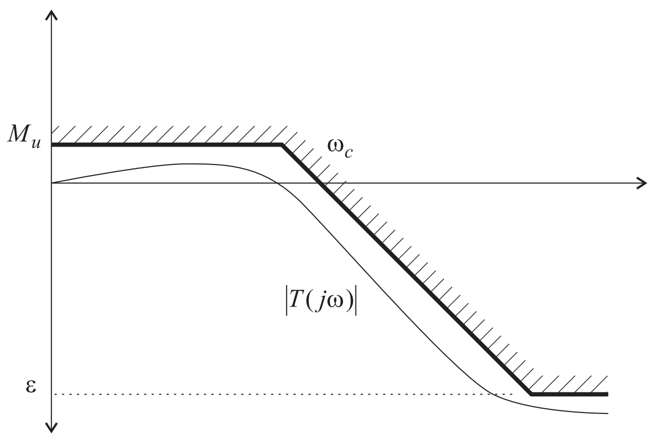
and applied to S



Similarly a good simple practical weighting function for the transmission T is

$$W_u = \frac{s + \omega_c/M_u}{\epsilon s + \omega_c}$$

and applied to T



In Matlab, the weights are labelled as

Direction	Text	Matlab	Parameter
$d_o \to y$	W_e	W_1	$S_o = (I + PK)^{-1}$
$u \to y$	W_u	W_2	$R = P(I + PK)^{-1}$
$r \to y$	W_y	W_3	$T = PK(I + PK)^{-1}$

We will be using weights primarily in the mixed sensitivity model

$$\left\| \begin{bmatrix} W_y S_o W_d \\ \rho W_1 T_o W_2 \end{bmatrix} \right\|_\infty$$

Tasks

1. Problem 5.4 on page 77: Let $G(s) = \frac{s-1}{(s+2)(s-3)}$. Find a stable coprime factorization $G = n(s)/m(s)$ such that $x\,n + y\,m = 1$.

2. Generate a suitable first order sensitivity bound weight W_s for the plant $P(s) = 1/s(s+1)$ and controller $K = 5$.