Robust Control Systems — Cross-Coupling

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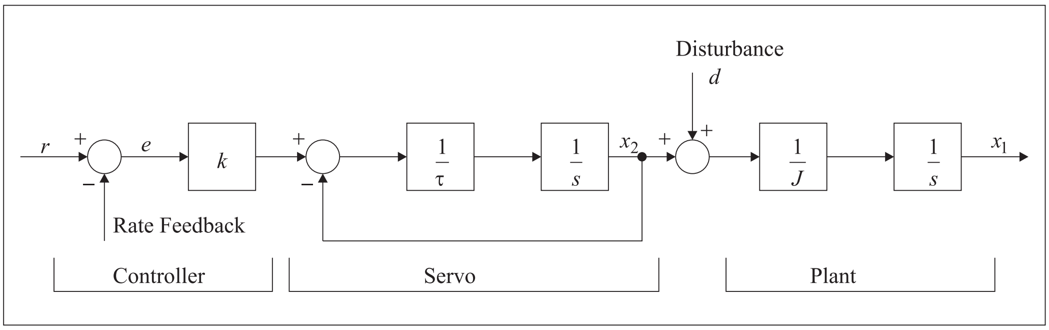
Consider the following example of a two-axis rate controller (for an aircraft of some sort). At first we assume that the axes are fully decoupled so we can design the controllers separately and independently. The model for any axis using a servo with time constant τ_1 is shown below.



For this open-loop system the state equations and transfer functions are

$$\begin{aligned} \dot{x}_1 &= \frac{1}{J}d + \frac{1}{J}x_2 \\ \dot{x}_2 &= \frac{k}{\tau}r - \frac{1}{\tau}x_2 \\ v &= x_1 \end{aligned}$$

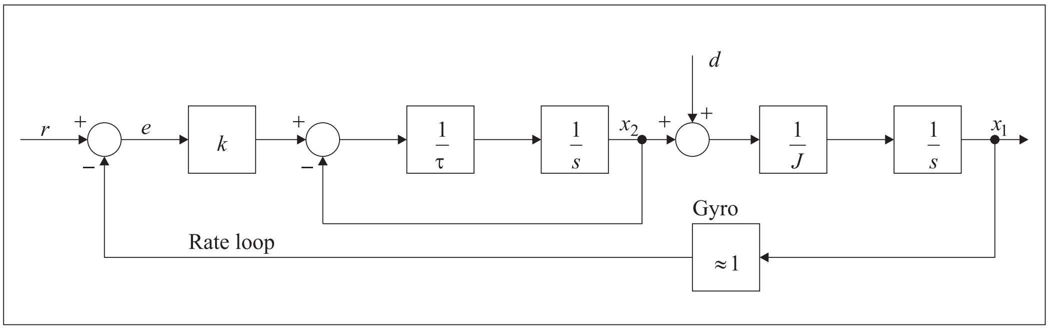
or in matrix form

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + Bu \\ v &= C\mathbf{x} \end{aligned}$$

with

$$A = \begin{bmatrix} 0 & \frac{1}{J} \\ 0 & -\frac{1}{\tau} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{J} & 0 \\ 0 & \frac{k}{\tau} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad u = \begin{bmatrix} d \\ r \end{bmatrix}$$

For each axis we use a inertial rate loop using a gyro as rate measurement device to stabilize each axis



In this case the feedback matrix K becomes

$$K = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

giving the closed loop dynamics

$$A - BK = \begin{bmatrix} 0 & \frac{1}{J} \\ 0 & -\frac{1}{\tau} \end{bmatrix} - \begin{bmatrix} \frac{1}{J} & 0 \\ 0 & \frac{k}{\tau} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{J} \\ -\frac{k}{\tau} & -\frac{1}{\tau} \end{bmatrix}$$

with the closed loop characteristic equation

$$s^2 + \frac{1}{\tau} s + \frac{k}{J\tau} = 0$$

If our servo has a time constant $\tau = 1/60 = 15\,\text{msec}$ and we want to achieve a $5\,\text{Hz}$ ($\approx 30\,\text{rad/s}$) rate loop bandwidth, we can calculate values of

$$\zeta \approx 1, \quad \frac{k}{J} = 15, \quad s^2 + 60s + 900 = (s+30)^2 = 0$$

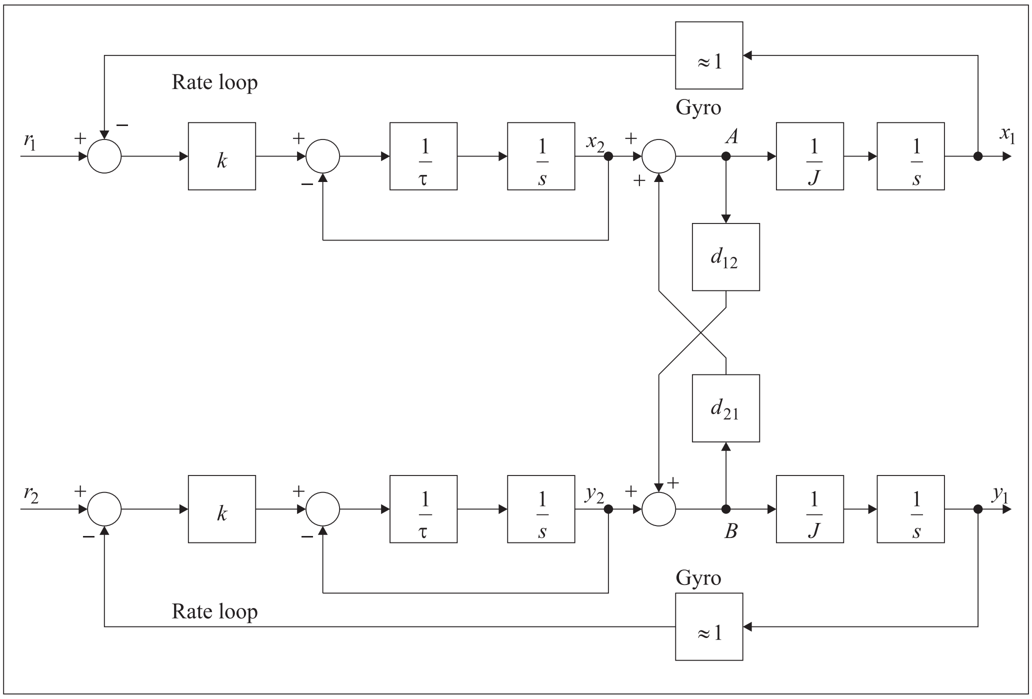
With these values the transfer functions from r and d to the output rate x_1 is

$$\begin{aligned} \frac{x_1}{r} &= \frac{\frac{k}{J\tau}}{s^2 + \frac{1}{\tau} s + \frac{k}{J\tau}} = \frac{900}{s^2 + 60s + 900} \\ \frac{x_1}{d} &= \frac{\frac{s + \frac{1}{\tau}}{J}}{s^2 + \frac{1}{\tau} s + \frac{k}{J\tau}} = \frac{15}{k} \frac{s + 60}{s^2 + 60s + 900} \end{aligned}$$

The static disturbance rejection looks good and is controlled by our gain function k

$$\frac{x_1/d}{x_1/r} = \frac{1}{k}$$

What happens when there are cross-coupling between the axes? In the following example the cross-coupling takes the form of a torque (or force) and acts from or on the torque summation.



The signals at the nodes A and B are

$$\begin{aligned}
A &= x_2 + d_{21}B \\
B &= y_2 + d_{12}A \\
\text{so } A &= x_2 + d_{21}y_2 + d_{21}d_{12}A, \quad \rightarrow \quad A = \frac{x_2 + d_{21}y_2}{1 - d_{21}d_{12}} \\
B &= y_2 + d_{12}x_2 + d_{12}d_{21}B, \quad \rightarrow \quad B = \frac{y_2 + d_{12}x_2}{1 - d_{12}d_{21}}
\end{aligned}$$

Just by inspection, we can see that the signals at nodes A and B will become very large if the product $d_{12}d_{21}$ approaches unity. The terms d_{12} and d_{21} are for this case scalars, therefore define the term

$$\alpha = \frac{1}{1 - d_{21}d_{12}} = \frac{1}{1 - d_{12}d_{21}}$$

For this open-loop system the 1st-order differential equations become

$$\dot{x}_1 = \frac{\alpha}{J}\left(x_2 + d_{21}y_2\right)$$

$$
\begin{aligned}
\dot{x}_2 &= \frac{k}{\tau}\, r_1 - \frac{1}{\tau}\, x_2 \\
\dot{y}_1 &= \frac{\alpha}{J}\,(y_2 + d_{12} x_2) \\
\dot{y}_2 &= \frac{k}{\tau}\, r_2 - \frac{1}{\tau}\, y_2
\end{aligned}
$$

giving state matrices

$$
A = \begin{bmatrix} 0 & \frac{\alpha}{J} & 0 & \frac{\alpha d_{21}}{J} \\ 0 & -\frac{1}{\tau} & 0 & 0 \\ 0 & \frac{\alpha d_{12}}{J} & 0 & \frac{\alpha}{J} \\ 0 & 0 & 0 & -\frac{1}{\tau} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ \frac{k}{\tau} & 0 \\ 0 & 0 \\ 0 & \frac{k}{\tau} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad u = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}
$$

With feedback K given by

$$
K = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}
$$

the system matrix with feedback becomes

$$
A - BK = \begin{bmatrix} 0 & \frac{\alpha}{J} & 0 & \frac{\alpha d_{21}}{J} \\ -\frac{k}{\tau} & -\frac{1}{\tau} & 0 & 0 \\ 0 & \frac{\alpha d_{12}}{J} & 0 & \frac{\alpha}{J} \\ 0 & 0 & -\frac{k}{\tau} & -\frac{1}{\tau} \end{bmatrix}
$$

The characteristic equation becomes

$$
s^4 + \frac{2}{\tau}\, s^3 + \left(\frac{1}{\tau^2} + \frac{2\alpha\, k}{\tau\, J}\right) s^2 + \frac{2\alpha\, k}{\tau^2\, J}\, s + \left(\frac{\alpha\, k}{\tau\, J}\right)(1 - d_{12}\, d_{21})
$$

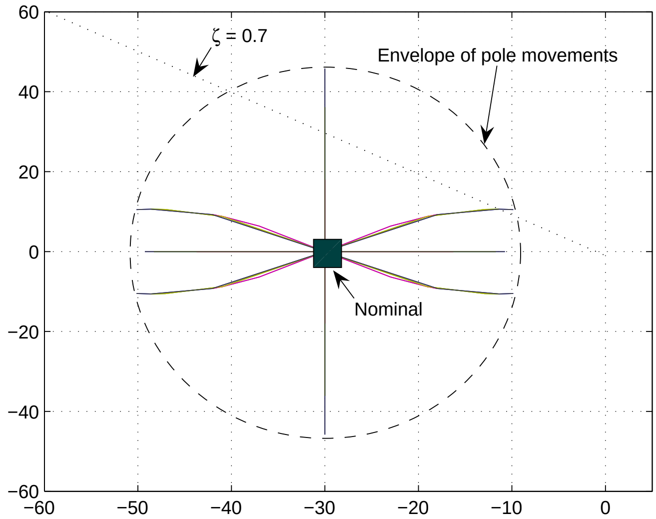
with closed loop poles at

$$
s = -\frac{1}{2\,\tau}\left[1 \pm \sqrt{1 - \frac{4\, k\, \tau\, \alpha(1 \pm \sqrt{d_{12}\, d_{21}})}{J}}\right]
$$

Test: With no cross-coupling, $d_{12} = d_{21} = 0$ we get $\alpha = 1$ and with $\tau = 1/60$ and $k/J = 15$ the characteristic equation becomes

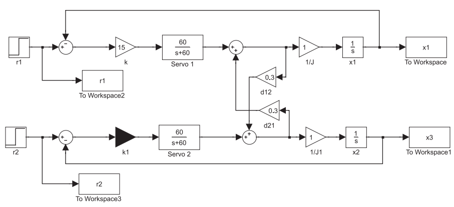
$$
s^4 + 120 s^3 + 5400\, s^2 + 108000\, s + 810000 = (s + 30)^4
$$

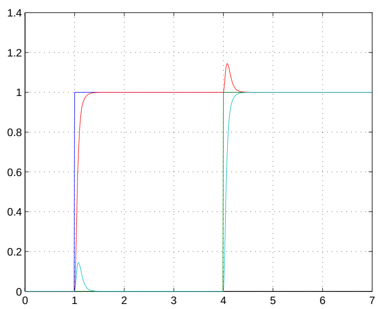
To see what the effect on the closed-loop poles will be, the coupling parameters d_{12} and d_{21} are varied in the range $-0.7 \le d_{xy} \le 0.7$. This creates the locus of closed loop pole movements as shown in the following diagram.



Although the system stays stable, the closed loop pole frequencies and damping range quite dramatically.

The step response shows that for a coupling of 0.3 a disturbance response of up to 20% can be expected.





Question The question posed by this example is: how should we go about designing control systems that may be influenced by known and unknown errors (modelling as well as disturbance) and is there any method of controlling the performance bounds?