Multivariable Systems I

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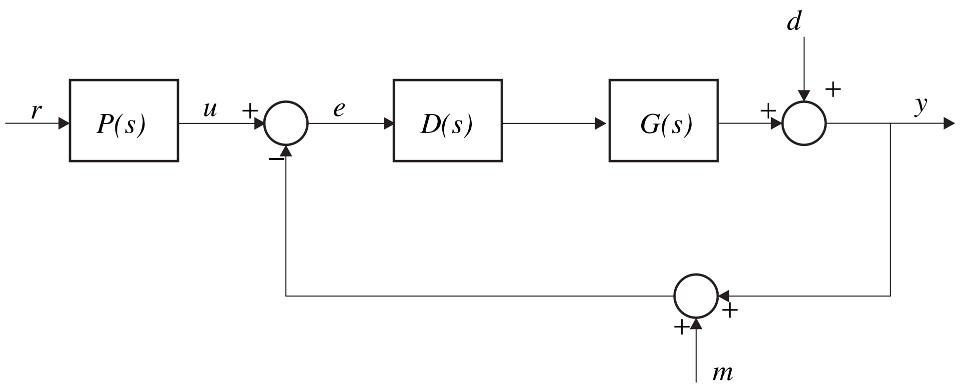
1 Single Loop Systems

From the block diagram



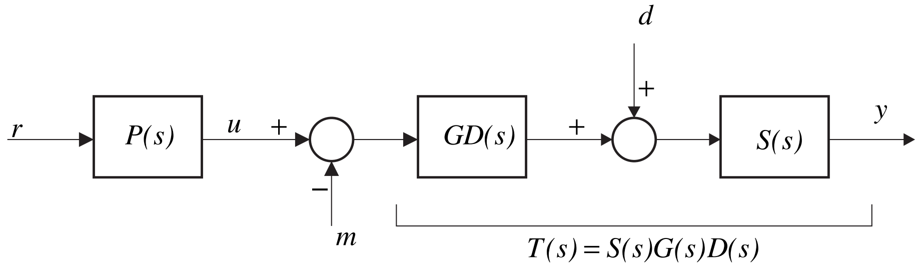
we get the equations

$$
\begin{aligned}
y(s) &= d(s) + G(s)D(s)\left[P(s)r(s) - m(s) - y(s)\right] \\
\left[I + GD(s)\right] y(s) &= d(s) + GD(s)\left[P(s)r(s) - m(s)\right] \\
y(s) &= \left[I + GD(s)\right]^{-1} d(s) + \left[I + GD(s)\right]^{-1} GD(s)\left[P(s)r(s) - m(s)\right] \\
y(s) &= S(s)d(s) + G_c(s)r(s) - T(s)m(s)
\end{aligned}
$$

where

$F_o(s)$	$= I + GD(s)$		Output Return Difference
$F_i(s)$	$= I + DG(s)$		Input Return Difference
$S(s)$	$= \left[I + GD(s)\right]^{-1}$	$= F_o^{-1}$	Sensitivity Function
$T(s)$	$= \left[I + GD(s)\right]^{-1} GD(s)$	$= S(s)GD(s)$	Closed Loop Transfer Function
$G_c(s)$	$= S(s)GD(s)P(s)$	$= T(s)P(s)$	Tracking Transfer Function

We can now simplify the diagram to



The relationship $T(s) + S(s)$ is given by

$$\begin{aligned} T(s) + S(s) &= [I + GD(s)]^{-1} GD(s) + [I + GD(s)]^{-1} \\ &= [I + GD(s)]^{-1} (GD(s) + I) \\ &= I \end{aligned}$$

To limit the effect of the disturbance $d(s)$ on the system, it is necessary to make $S(s)$ very small — this requires us to make $GD(s)$ very large. Under these circumstances $T(s)$ approaches unity. If we want make $T(s)$ small, then $S(s)$ must be close to unity and the effect of $d(s)$ is large.

The usual strategy to overcome this conflict is to make $S(s)$ small at low frequencies and to make $T(s)$ small at high frequencies.

To make $S(s)$ small at low frequencies up to ω_0

$$|S(j\omega)| < \epsilon, \quad 0 \leq \omega \leq \omega_0$$

we must make $|G(j\omega)K(j\omega)|$ large. If $|G(j\omega)|$ is not large, then we must make $|K(j\omega)|$ large for $\omega < \omega_0$. Also, approximately

$$|G(j\omega)K(j\omega)| > 1/\epsilon, \quad 0 \leq \omega \leq \omega_0$$

Also

$$|T(j\omega)| = |1 - S(j\omega)| \approx 1, \quad 0 \leq \omega \leq \omega_0$$

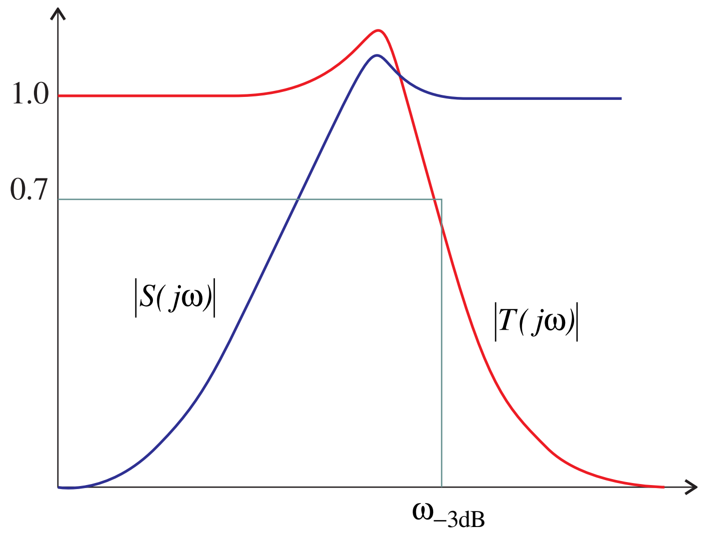
and $T(0) \simeq 1$. This dictates the shape of $T(j\omega)$ in in the figure below.

To make $T(s)$ small at high frequencies as $\omega \to \infty$

$$|T(j\omega)| < \delta, \quad \omega_0 \leq \omega \leq \infty$$

we must make $|G(j\omega)K(j\omega)|$ small. If $|G(j\omega)|$ is not small, then we must make $|K(j\omega)|$ small for $\omega > \omega_0$.

So the normal solution is: High $T(j\omega)$ at low frequencies and high $S(j\omega)$ at high frequencies.



2 Design Approaches

2.1 Closed Loop Design

The one approach is to calculate the controller transfer function from the closed loop description $T(s) = S(s)G(s)D(s)$

$$D(s) = G^{-1}(s)S^{-1}(s)T(s) = G^{-1}(s)F_0(s)T(s)$$

This approach leads to serious trade-offs to make $|S(j\omega)|$ small and $|T(j\omega)|$ small.

2.2 Open Loop Design

The common method is to work with the 'open-loop' return ratio $G(s)D(s)$ directly. The design criteria is typically

- $G(j\omega)D(j\omega) > L \geq 1$, for $0 \leq \omega \leq \omega_0$.
- $G(j\omega)D(j\omega) < \delta \leq 1$, for $\omega_0 \leq \omega \leq \infty$.
- Gain margin $> \mu$, phase margin $> \phi$
- Locus of $G(j\omega)D(j\omega)$ stay outside the -1 point with distance given by M-circle in Nichols chart.

3 Performance Limitations

3.1 Gain–Phase Relationships

With a transfer function $Q(s)$ which is minimum phase (no right-half plane zeroes), the gain and phase is given by

$$\begin{aligned}
|L(j\omega)| &= \log|Q(j\omega)| \\
\phi(j\omega) &= \arg Q(j\omega) \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dL(\omega)}{d\omega}\log\coth|\frac{\omega}{2}|\,d\omega \\
&\approx \frac{\pi}{2}\frac{dL}{d\omega}
\end{aligned}$$

This is the important Bode plot rule of thumb

- If the gain falls at 20 dB per decade, then the phase will approach $-90°$.
- If the gain falls at 40 dB per decade, then the phase will approach $-180°$.

Around the 0 dB point we would restrict the fall of gain to 20 dB to achieve an adequate phase margin.

This requirement restricts the design specifications achievable eg. it is not possible to achieve a stable design with gain of 30 dB at 1 rad/s and -30 dB at 10 rad/s — the average slope of the gain would be -60 dB per decade with a phase approaching $-270°$.

3.2 Right Half-Plane Zeroes

With a function containing a real right half-plane zero, we can write

$$G(s) = G_1(s)A(s) = G_1(s)\frac{s-a}{s+a}$$

with the last function $A(s)$ the familiar all-pass function (Gain = 1 but phase decreases). The phase of $A(s)$ is given by

$$\angle A(j\omega) = -\arctan(\omega/a) - \arctan(\omega/a) = -2\arctan(\omega/a)$$

At the cross-over frequency $\omega = a$, then $\angle A(ja) = -\pi/2$. This leaves only typically 45°– 60° phase margin for the control function itself.

The implication is that right-half-plane zeroes limit the available bandwidth to that of the position of the right-half-plane zero.

3.3 Right Half-Plane Poles

With a function containing a real right half-plane pole, we can write

$$G(s) = G_1(s)A^{-1}(s) = G_1(s)\frac{s+a}{s-a}$$

with the last function $A(s)$ the familiar all-pass function. Assume that $G_1(j\omega)$ is monotonic decreasing with ω.

Let the phase margin of $\angle G_1(j\omega_c) = \phi$. Then

$$\begin{aligned} \angle G(j\omega_c) &= \angle G_1(j\omega_c) + \angle A^{-1}(j\omega_c) \\ &< \angle A^{-1}(j\omega_c) - \pi \end{aligned}$$

therefore

$$\angle A^{-1}(j\omega_c) > \phi + \pi$$

If $\phi = \pi/4$, then $\omega_c > a/2$, and if $\phi = \pi/2$, then $\omega_c > a$.

The bandwidth of the closed loop must be approximately $\omega_c \approx a$. Such a control system has high bandwidth (agile) but with poor noise filtering characteristics.

3.4 Bode's Integral Theorem

Let the sensitivity function $S(s) = [1 + G(s)K(s)]^{-1}$. If the closed loop is stable, but with N right half-plane poles $\sum_{i=1}^{N} p_i$

$$\int_0^\infty \ln|S(j\omega)|\,d\omega = \pi\sum_{i=1}^{N}\Re(p_i)$$

Two aspects are

1. If the system is stable

   $$\int_0^\infty \ln|S(j\omega)|\,d\omega = 0$$

   This implies that there is as much sensitivity increase as decrease — when plotted in dB, the positive area will equal the negative area. If the sensitivity is decreased at lower frequency, a corresponding increase at higher frequency will result. If

$$|G(j\omega)| < \frac{M}{\omega^{1+k}} < \epsilon, \omega > \omega_c$$

then

$$\int_{\omega_c}^{\infty} \ln |S(j\omega)| \, d\omega < \frac{\omega_c}{k} \ln \frac{1}{1-\epsilon}$$

which tends to zero as ϵ tend to zero, so most of the positive area must be gained below ω_c if ϵ is very small.

2. For open loop unstable systems, there is more positive than negative area. This means that part of the feedback is used to stabilise the system — and only part is used to reduce sensitivity.

So far we have used the controller and plant trasfer functions directly to calculate the Sensitivity, transfer function, return differences directly. This is not very convenient as these functions are typically very frequency dependent and difficult to interpret beyond the simple cases. By using norms and bounds instead of transfer functions, the problem is simplified.

4 Definition of Operator Norms

The gain of a SISO (single input-single output) system can be expressed as a transfer function:

$$G(s) = k\frac{(s+a)}{(s+b)(s+c)}$$

The above function expresses the gain G as a function of the complex frequency variable s. The singular points on this gain occur at the frequencies $s = -c, -b, -a$. At the first two the gain is infinite (poles) and at the last one the gain is zero.

If we extend this concept to multivariable systems we pick up a problem: what is the gain of a matrix. One definition that has been accepted is the norm of the matrix

$$||G(s)||$$

Let X be a vector space, then a real-valued function $||\cdot||$ is a norm on X if it satisfies:

1. $||x|| \geq 0$ i.e. positive semi-definite, as $||x|| = 0$ if and only if $x = 0$.

2. $||\alpha x|| = |\alpha| \, ||x||$

3. $||x+y|| \leq ||x|| + ||y||$

Define now the vector p-norm as

$$||x||_p := \left(\sum_{i=1}^{N} |x_i|^p \right)^{\frac{1}{p}}$$

When $p = 1, 2, \infty$ we get

$$\begin{aligned} ||x||_1 &= \sum_{i=1}^{N} |x_i| \\ ||x||_2 &= \sqrt{\sum_{i=1}^{N} |x_i|^2} \\ ||x||_\infty &= \max_{1 \leq i \leq n} |x_i| \end{aligned}$$

A vector norm is essentially the *length* of the vector.

A matrix norm induced by a vector norm is defined by

$$||A||_p := \sup_{x \neq 0} \frac{||Ax||_p}{||x||_p}$$

with $\sup_{x \neq 0}$ as the maximum value. The matrix norms map a vector from one space to another — one can interpret it as a *gain*. Some examples:

- The induced matrix 2-norm or euclidean vector norm $||x|| = \sqrt{x^T x}$ is for example
$$||A||_2 = \sqrt{\lambda_{\max}(A^T A)} = \overline{\sigma}$$
where $\overline{\sigma}^2$ is the maximum eigenvalue of $A^T A$ (or AA^T).

Let $G(s)$ be a proper transfer function with no poles on the imaginary axis. Then define

$$\begin{aligned} ||G(s)||_2 &= \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{trace}[G(j\omega) G^T(-j\omega)] \, d\omega} \\ ||G(s)||_\infty &= \sup_{\omega} \overline{\sigma}[G(j\omega)] \qquad \textit{maximum upper bound of } G(s) \end{aligned}$$

The following properties are important

$$\begin{aligned} ||G(s)||_2^2 &= \mathcal{E}\{y^T y\} \\ ||G(s)||_\infty &= \sup_\omega \frac{||y||_2}{||u||_2} \end{aligned}$$

Clearly $||G(s)||_2$ defines the power gain of the $G(s)$ if the input is a white noise process — compare with Kalman filter formulation.

5 Hilbert and Hardy spaces

Note: Analytic functions have all derivatives over the region in question (we can generate a power series).

Hilbert space $\mathcal{L}_2$: A Hilbert space is defined by the inner product space with norm of

$$\mathcal{L}_2 := \int_a^b \text{trace}[f(t) * g(t)]\, dt$$

If we map the imaginary axis on our function then we get the following Hilbert space $\mathcal{L}_2(jR)$ with

$$\mathcal{L}_2[F(j\omega)] = \int_\infty^{-\infty} \text{trace}[F^*(j\omega) * F(j\omega)]\, d\omega < \infty$$

then the inner product induced norm is defined by

$$||F||_2 := \sqrt{< F, F >}, \quad < F, G >:= \frac{1}{2\pi} \int_\infty^{-\infty} \text{trace}[F^*(j\omega) * G(j\omega)]\, d\omega$$

Hardy space $\mathcal{H}_2$: $\mathcal{H}_2$ is a closed subspace of $\mathcal{L}_2(jR)$ with matrix functions $F(s)$ analytic in $\text{Re}(s) > 0$ with norm

$$||F||_2^2 := \frac{1}{2\pi} \int_\infty^{-\infty} \text{trace}[F^*(j\omega) * F(j\omega)]\, d\omega$$

Hardy space $\mathcal{H}_\infty$: $\mathcal{H}_\infty$ is a subspace of $\mathcal{L}_\infty$ with functions that are analytic and bounded in the open right-half plane

$$||F||_\infty := \sup \overline{\sigma}[F(s)]$$

The Hardy space defines a bounded function, so the $\mathcal{H}_\infty$ norm is bounded over all frequencies. This means that $F(s)$ must be stable and strictly proper (more finite poles than zeroes).

6 Calculating Norms of Systems

Given a system

$$G(s) = C(sI - A)^{-1}B + D, \qquad \begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{cases}$$

then by calculating the controllability (or observability) grammian Q which can be obtained from the Lyapunov equations (Matlab `gram(A,B)` or `lyap(A,B*B')`)

$$A^T Q + QA + C^T C = 0 \quad \text{or} \quad AP + PA^T + B^T B = 0$$

we may compute the $\mathcal{H}_2$ norm from

$$||G||_2^2 = \text{trace}(B^T QB) = \text{trace}(CPC^T)$$

Given a rational transfer function $G(s) = G(s)_+ + G(s)_-$, we may split the calculation as shown in the following example.

Example Consider the transfer matrix with stable and unstable poles

$$G(s) = \begin{bmatrix} \frac{3(s+3)}{(s-1)(s+2)} & \frac{2}{(s-1)} \\ \frac{(s+1)}{(s+2)(s+3)} & \frac{1}{(s-4)} \end{bmatrix}$$

Split the transfer matrix in its stable and unstable transfer matrices

$$G(s) = G_s(s) + G_u(s)$$

The stable common roots are $\{-2, \ -3\}$ and the unstable roots are $\{+1, \ +4\}$. The terms are then using the matlab function `sdecomp`

$$G_s(s) = \left[\begin{array}{cc|cc} -2 & 0 & -1 & 0 \\ 0 & -3 & 2 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right], \quad G_u(s) = \left[\begin{array}{cc|cc} 1 & 0 & 4 & 2 \\ 0 & 4 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

Then the command `h2norm(Gs)` gives $||G_s||_2 = 0.6055$ and `h2norm(cjt(Gu))` gives $||G_u||_2 = 3.182$, so we have $||G||_2 = \sqrt{||G_s||_2^2 + ||G_u||_2^2} = 3.23$.

The value of $||G||_\infty$ may be read off the Bode plot. The calculation for our system

$$G(s) = C(sI - A)^{-1}B + D$$

involves a operator $\gamma > 0$ such that $||G||_\infty < \gamma$.

We compute the smallest value of γ such that the Hamiltonian matrix

$$H = \left[\begin{array}{cc} A + BR^{-1}D^TC & BR^{-1}B^T \\ -C^T(I + DR^{-1}D^T)C & -(A + BR^{-1}D^TC)^T \end{array} \right]$$

has no eigenvalues on the imaginary axis, where $R = \gamma^2 I - D^T D$. This normally an iterative procedure, starting with a large γ. A bisection algorithm to perform the search for a suitable γ is typically used.

To understand the difference between the two norms, write the $\mathcal{H}_2$ norm in terms of singular values

$$||G||_2^2 = \frac{1}{2\pi} \int_{\infty}^{-\infty} \sum_i \sigma_i^2(G(j\omega))\, d\omega$$

The $\mathcal{H}_2$ norm therefore tries to minimize the sum of the square of average singular values over all frequencies. To summarize

- $\mathcal{H}_2$: Push the envelope down based on "average" of singular values
- $\mathcal{H}_\infty$: Push the envelope down based on the peak singular value

Example Consider the simple plant

$$G(s) = \frac{1}{s + a}$$

The $\mathcal{H}_2$ norm is

$$||G||_2 = \sqrt{\frac{1}{2\pi} \int_{\infty}^{-\infty} \frac{1}{\omega^2 + a^2}\, d\omega} = \sqrt{\frac{1}{2a}}$$

and the $\mathcal{H}_\infty$ norm is

$$||G||_\infty = \max_\omega \sqrt{\frac{1}{\omega^2 + a^2}} = \frac{1}{a}$$

Why is the $\mathcal{H}_\infty$ norm so popular? It is an induced norm and satisfies the multiplicative property

$$||A(s)B(s)||_\infty \leq ||A(s)||_\infty \cdot ||B(s)||_\infty$$

which is very convenient for handling unstructured model uncertainty.

The $\mathcal{H}_2$ norm is not an induced norm and does not satisfy this property.

Example From the previous function, the $\mathcal{H}_2$ norm of $G^2(s)$ is

$$||G^2(s)||_2 = \sqrt{\frac{1}{a}}\ ||G(s)||_2 > ||G(s)||_2 \cdot ||G(s)||_2 \text{ for } a < 1$$

whereas

$$||G^2(s)||_\infty = \frac{1}{a^2} = ||G(s)||_\infty \cdot ||G(s)||_\infty$$

The important `Matlab 6+` functions here are:

- `G = ss(A,B,C,D)` to enter the system
- `normhinf(G,0.0001)` or `normh2(G,0.0001)` for 0.0001 relative error (default 0.001).

7 Principle Gains

The symbols $\overline{\sigma}$ and $\underline{\sigma}$ are defined as follows

$$\begin{aligned}\overline{\sigma}(G) &= \max_{||x||=1} ||Gx|| \equiv \sqrt{\lambda_{\max}[G^*G]} = \sup_{x\neq 0} \frac{||Gx||}{||x||} \\ \underline{\sigma}(G) &= \min_{||x||=1} ||Gx|| \equiv \sqrt{\lambda_{\min}[G^*G]}\end{aligned}$$

The quantities $\overline{\sigma}$ and $\underline{\sigma}$ are called the maximum and minimum singular gains or principle gains of G respectively.

Note that $\overline{\sigma}[G(j\omega)] = ||G(j\omega||_s$ which changes with frequency.

A useful characteristic of a system is its condition number which is defined as

$$\text{cond}[G(j\omega)] = \frac{\overline{\sigma}[G(j\omega)]}{\underline{\sigma}[G(j\omega)]}$$

It is sometimes used to measure the difficulty of inverting a matrix, and it gives some indication of the difficulty of controlling the plant.

The gain matrix $G(s)$ can be written as

$$G = Y\Sigma U^* \quad \text{with} \quad \Sigma = \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_m\}$$

This is known as singular decomposition — one of the key elements inside the Matlab algorithms.

The columns of U represent the output directions of $G(s)$ and the columns of V represent the input directions. Both are of unit length and orthogonal. We may now write

$$Gv_i = \sigma_i u_i$$

which describes the gain of the system from in the direction i. Now

$$G^{-1} = [Y \Sigma U^*]^{-1} = U \Sigma^{-1} Y^*$$

so

$$||G^{-1}(j\omega)||_s = \frac{1}{\underline{\sigma}(\omega)}$$

therefore

$$\underline{\sigma}(\omega) \leq ||G(j\omega)|| \leq \overline{\sigma}(\omega)$$

which means that the gain of the multivariable system is sandwiched between the smallest and largest principal gains.

Example With

$$G = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$$

the singular value decomposition of G (using `Matlab` function `svd`) is

$$G = \begin{bmatrix} 0.872 & 0.490 \\ 0.490 & -0.872 \end{bmatrix} \begin{bmatrix} 7.343 & 0 \\ 0 & 0.272 \end{bmatrix} \begin{bmatrix} 0.794 & -0.608 \\ 0.608 & 0.794 \end{bmatrix}$$

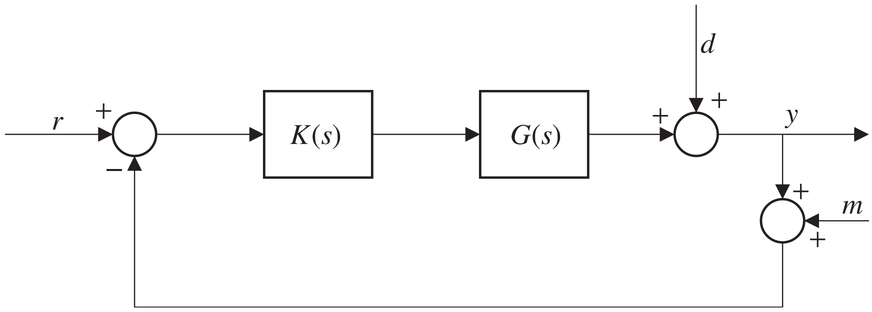
The largest gain $\overline{\sigma} = 7.343$ is for an input in the direction $\overline{v} = \begin{bmatrix} 0.794 \\ 0.608 \end{bmatrix}$ and the smallest gain $\underline{\sigma} = 0.272$ is for an input in the direction $\underline{v} = \begin{bmatrix} -0.608 \\ 0.794 \end{bmatrix}$. As both inputs affect both outputs, the system is clearly interactive. The condition number cond$[G] = \frac{\overline{\sigma}}{\underline{\sigma}} = 27$, showing that the system is ill-conditioned.

Note that a matrix is positive definite if $x^T Ax > 0$ for all $x \neq 0$. Similarly a matrix is positive semidefinite if $x^T Ax \geq 0$ for all $x \neq 0$ (i.e. some values of x may provide a zero solution). In this case the rank of the matrix will be smaller than the matrix order.

8 Use of Principle Gains

In the case of the SISO system expressed as a standard feedback configuration in the figure below, the sensitivity is obtained from the return difference

$$S(j\omega) = [1 + G(j\omega)K(j\omega)]^{-1}$$



The normal requirement is to have a high sensitivity for all frequencies where the command, disturbances and/or plant changes.

This idea extended to MIMO systems requires

$$\overline{\sigma}\left[[I + G(j\omega)K(j\omega)]^{-1}\right] \quad \text{small}$$

or conversely

$$\underline{\sigma}\,[I + G(j\omega)K(j\omega)] \quad \text{large}$$

Example:

$$A = \begin{bmatrix} -3.0 & -0.1 \\ -0.2 & -4.0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0.1 \\ 0.2 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

In this case we can estimate the pole positions around $s = -3$ and $s = -4$ with radii of uncertainty of 0.1–0.2. The exact pole positions are at $s = -2.98, -4.02$.

The four scalar transfer functions are

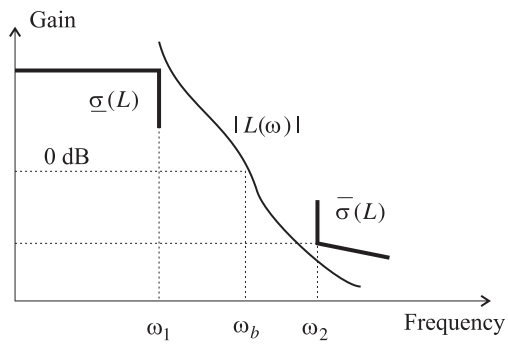
$$\begin{aligned}
\frac{y_1}{u_1} &= \frac{s + 3.98}{s^2 + 7s + 11.98} \\
\frac{y_2}{u_1} &= 0.2\frac{s + 2}{s^2 + 7s + 11.98} \\
\frac{y_1}{u_2} &= 0.1\frac{s + 3}{s^2 + 7s + 11.98} \\
\frac{y_2}{u_2} &= \frac{s + 2.98}{s^2 + 7s + 11.98}
\end{aligned}$$

The norm of A, i.e. $\overline{\sigma}(A) = 4.02$ — see `Matlab` function `norm(A)`. The singular value decomposition function `svd` produces

$$A = USV \quad U = \begin{bmatrix} -0.14 & -0.99 \\ -0.99 & 0.14 \end{bmatrix} \quad S = \begin{bmatrix} 4.02 & 0 \\ 0 & 2.98 \end{bmatrix} \quad V = \begin{bmatrix} 0.15 & 0.99 \\ 0.99 & -0.15 \end{bmatrix}$$

with the diagonal elements of S the principle gains in descending order.

How do we use these svd values? The setting of specifications is a very important part of multivariable design — see the diagram below where we limit the low-frequency gain for performance $\underline{\sigma}(L)$ and the high-frequency gain for robustness $\overline{\sigma}(L)$.



Example: With $G(s)$ as given in the previous system and with K given by

$$K = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

and then $GK(s)$ is

$$GK_{11} = 2\frac{s + 3.98}{s^2 + 7s + 11.98} \qquad GK_{12} = 0.4\frac{s + 2}{s^2 + 7s + 11.98}$$

$$GK_{21} = 0.1\frac{s + 3}{s^2 + 7s + 11.98} \qquad GK_{22} = \frac{s + 2.98}{s^2 + 7s + 11.98}$$

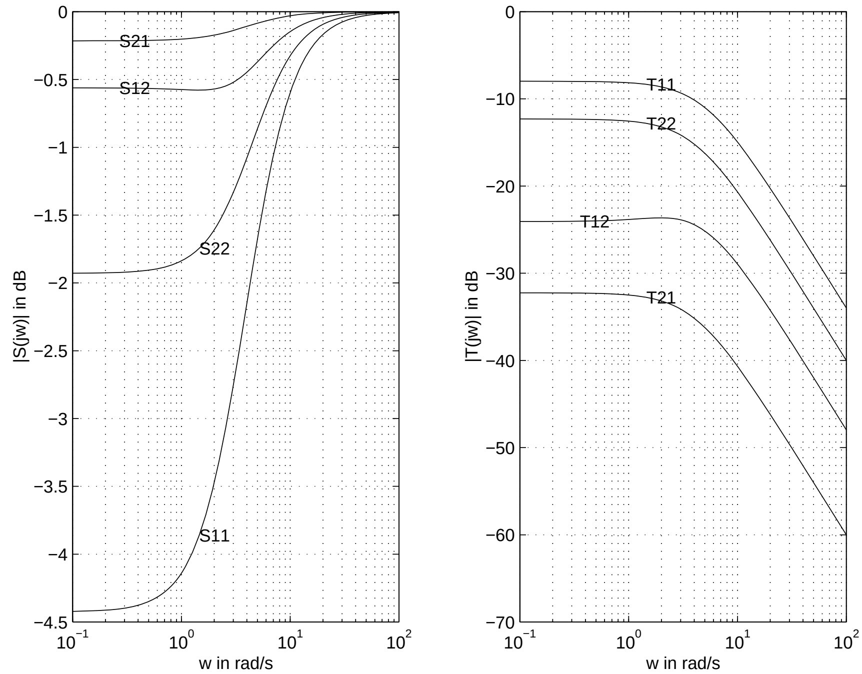
and then $S(s) = [I + GK(s)]^{-1}$ is

$$S_{11}(s) = \frac{s^2 + 7s + 11.98}{s^2 + 9s + 19.94} \qquad S_{12}(s) = \frac{s^2 + 7s + 11.98}{s^2 + 7.4s + 12.78}$$

$$S_{21}(s) = \frac{s^2 + 7s + 11.98}{s^2 + 7.1s + 12.28} \qquad S_{22}(s) = \frac{s^2 + 7s + 11.98}{s^2 + 8s + 14.96}$$

and $T(s) = GK(s)\left[I + GK(s)\right]^{-1}$ is

$$T_{11}(s) = 2\frac{s+3.98}{s^2+9s+19.94} \qquad T_{12}(s) = 0.4\frac{s+2}{s^2+7.4s+12.78}$$
$$T_{21}(s) = 0.1\frac{s+3}{s^2+7.1s+12.28} \qquad T_{22}(s) = \frac{s+2.98}{s^2+8s+14.96}$$



The major design goals, expressed in terms of the principle gains becomes:

Sensitivity: Keep $\overline{\sigma}\left[(I+GK)^{-1}\right]$ as small as possible — this implies $\underline{\sigma}[GK]$ large.

Noise Propagation: Keep $\overline{\sigma}\left[I+(I+GK)^{-1}\right]$ as small as possible — this implies $\overline{\sigma}[GK]$ small. This requirement is in conflict with the sensitivity requirement.

Tracking signal: Keep $\overline{\sigma}\left[I+(I+GK)^{-1}\right] \approx 1$ and $\underline{\sigma}\left[I+(I+GK)^{-1}\right] \approx 1$ — this implies $\underline{\sigma}[GK]$ large. This requirement is in conflict with the noise propagation requirement but not the sensitivity requirement.

Minimize Energy: Keep $\overline{\sigma}[K]$ as small as possible.

9 Packed Format — Section 3.1

A shorthand notation is used to describe the linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

which can be written in the following matrix form

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$

for which the following short-hand notation is used

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] :\equiv C(sI - A)^{-1}B + D$$

Note that

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \neq \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

The left-hand side is a transfer function and the right hand only an augmented matrix.

The observer-based controller with equation set

$$\begin{aligned} \dot{\hat{x}} &= (A - LC)\hat{x} + Bu + Ly \\ u &= -Kx \end{aligned}$$

has a transfer function

$$D(s) = \frac{u}{y}(s) = \left[\begin{array}{c|c} A - BK - LC & L \\ \hline -K & 0 \end{array}\right] = -K(sI - A + BK + LC)^{-1}L$$

Example 3.2 on page 33.

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

With the desired controller closed-loop poles at $\{-2,\ -3\}$ and observer poles at $\{-10,\ -10\}$ we find

$$K = \begin{bmatrix} -6 & -8 \end{bmatrix}, \quad L = \begin{bmatrix} -21 \\ -51 \end{bmatrix} \quad \rightarrow \quad D(s) = \frac{-534(s + 0.6966)}{(s + 34.6564)(s - 8.6564)}$$

which is definitely not robust.

10 Operations on Systems

The series connection of two systems

$$G_1(s) = \frac{y_1}{u_1}(s) = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right], \quad G_2(s) = \frac{y_2}{u_2}(s) = \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]$$

can be constructed using the connection equation

$$u_2 = y_1 = C_1 x_1 + D_1 u_1$$

so that the second output equation becomes

$$y_2 = C_2 x_2 + D_2 u_2 = C_2 x_2 + D_2 C_1 x_1 + D_2 D_1 u_1$$

Similarly the second state equation set becomes

$$\dot{x}_2 = A_2 x_2 + B_2 u_2 = A_2 x_2 + B_2 C_1 x_1 + B_2 D_1 u_1$$

so the combined equation set becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 D_1 \end{bmatrix} u_1$$

for the transfer function in shorthand notation

$$G(s) = G_2 G_1(s) = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ B_2 C_1 & A_2 & B_2 D_1 \\ \hline D_2 C_1 & C_2 & D_2 D_1 \end{array}\right]$$

The `Matlab` function `series` performs this function for us. Similarly the parallel combination is simply

$$G_1(s) + G_2 9s) = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array}\right]$$

The Matlab functions are `mmult, madd, append, series, parallel, feedback` etc.

11 Transfer Function or Polynomial Matrices of Multivariable Systems

When analysing multivariable systems, the standard approach is to use a polynomial matrix realization. This can be easily determined from the state-space (or time realization).

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Note that the number of columns in B, D may be more than 1 (multiple inputs) and the number of rows in C, D may be more than 1 (multiple outputs).

The poles of the $G(s)$ can be found from the eigenvalues of A (`Matlab` function `eig()`), and the transmission zeroes by solving the equation

$$\begin{bmatrix} A - s_0 I & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = 0$$

(typically `Matlab` function `tzero`). If $u = 0$, then $s = s_0$ is a uncontrollable mode.

Example

The transfer function

$$G(s) = \begin{bmatrix} \frac{s+1}{s+2} & \frac{s+3}{s+4} \end{bmatrix}$$

is a minimal realisation. The poles of the system are really $s = -2; -4$ and the obvious extra zeroes at

$$G(s) = \begin{bmatrix} \frac{(s+1)(s+4)}{(s+2)(s+4)} & \frac{(s+1)(s+3)}{(s+2)(s+4)} \end{bmatrix}$$

These extra modes are either uncontrollable or unobservable (depending on the inputs and outputs).

Example 3.3 on page 41.

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{ccc|ccc} -1 & -2 & 1 & 1 & 2 & 3 \\ 0 & 2 & -1 & 3 & 2 & 1 \\ -4 & -3 & -2 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 4 & 0 & 0 & 0 \end{array}\right]$$

Use the Matlab commands

```
G1 = pck(A,B,C,D); z0 = szeros(G1) % or
G2 = ss(A,B,C,D);  z2 = szeros(G1) % or
z1 = tzero(A,B,C,D)
```

which gives $z_0 = z_1 = z_2 = 0.2$.

We can also find y and v such that

$$\begin{bmatrix} x^T & v^T \end{bmatrix} \begin{bmatrix} A - zI & B \\ C & D \end{bmatrix} = 0 =$$

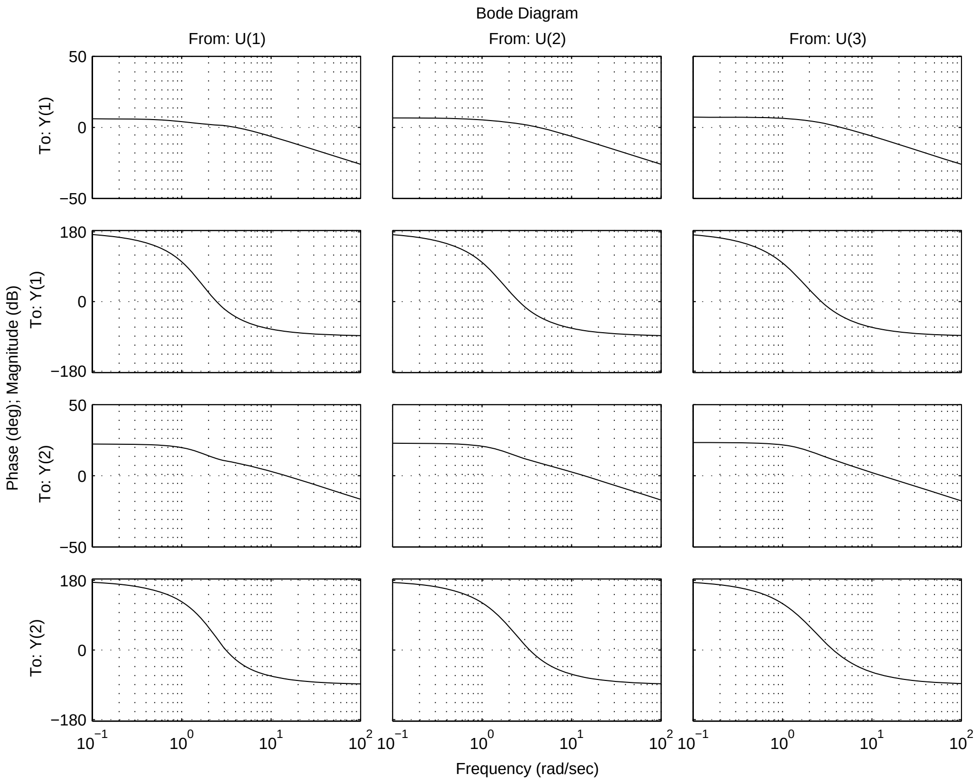
using the Matlab command `null([A - z0*eye(3),B;C,D]')` giving

$$\begin{bmatrix} y \\ \hline v \end{bmatrix} = \begin{bmatrix} 0.0466 \\ 0.0466 \\ -0.1866 \\ \hline -0.9702 \\ 0.1399 \end{bmatrix}$$

In this example the singular values using $S = \texttt{svd}(A)$ are

$$S = \begin{bmatrix} 5.7053 \\ 2.6911 \\ 0.4559 \end{bmatrix}$$

therefore $\overline{\sigma} = 5.7053$ and $\underline{\sigma} = 0.4559$. The eigenvalues are at $s = -1.5370 \pm j1.0064; 2.0739$ and the transmission zeroes at $s = -0.9702; 0.1399$. The Bode plot of G is shown here



Example 4.3 on page 60

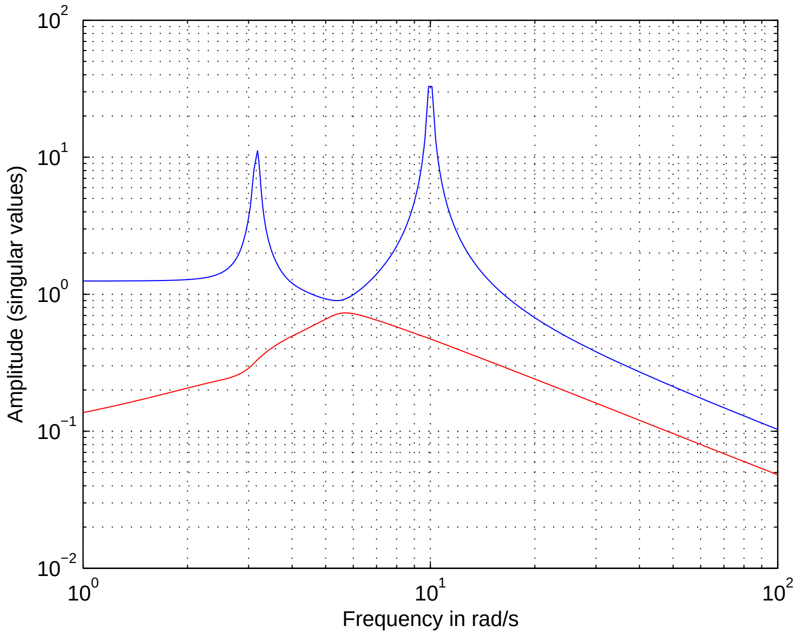
$$G(s) = \begin{bmatrix} \frac{10(s+1)}{s^2+0.2s+100} & \frac{1}{s+1} \\ \frac{s+2}{s^2+0.1s+10} & \frac{5(s+1)}{(s+2)(s+3)} \end{bmatrix}$$

Matlab script

```
G11 = nd2sys([10 10],[1 0.2 100]);
G12 = nd2sys(1,[1 1]);
G21 = nd2sys([1 2],[1 0.1 10]);
G22 = nd2sys([5 5],[1 5 6]);
G   = sbs(abv(G11,G21),abv(G12,G22));
   % abv - Place G11 and G21 on top of each other
   % sbs - put matrices next to each other
w  = logspace(0,2,200);
Gf = frsp(G,w);          % Compute frequency response
[u,s,v] = vsvd(Gf);      % SVD at all frequencies
vplot('liv,lm',s),grid   % Plot it
pkvnorm(s)               % Norm from the plot
```

```
hinfnorm(G,0.0001)       % Search for norm
```

with the singular values plots



The function `pkvnorm` suggests that $\overline{\sigma} = 32.86$ but the function `hinfnorm(G,0.0001)` gives a higher value of 50.25 with the bisection algorithm.

12 Homework Problems

3.7 Compute the system zeroes and corresponding zero directions of the following transfer functions:

$$G_1(s) = \left[\begin{array}{cc|cc} 1 & 2 & 2 & 2 \\ 1 & 0 & 3 & 4 \\ \hline 0 & 1 & 1 & 2 \\ 1 & 0 & 2 & 0 \end{array}\right], \quad G_2(s) = \left[\begin{array}{cc|cc} -1 & -2 & 2 & 2 \\ 0 & 1 & 2 & 1 \\ \hline 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right]$$

$$G_3(s) = \left[\begin{array}{cc} \frac{2(s+1)(s+2)}{s(s+3)(s+4)} & \frac{(s+2)}{(s+1)(s+3)} \end{array}\right], \quad G_4(s) = \left[\begin{array}{cc} \frac{1}{s+1} & \frac{s+3}{(s+1)(s-2)} \\ \frac{10}{s-2} & \frac{5}{s+3} \end{array}\right]$$

Also find the vectors x and u whenever appropriate so that either

$$\begin{bmatrix} A - zI & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = 0 \quad \text{or} \quad \begin{bmatrix} x^T & u^T \end{bmatrix} \begin{bmatrix} A - zI & B \\ C & D \end{bmatrix} = 0$$

4.4 Compute the 2-norm and ∞-norm of the following transfer systems:

$$G_1(s) = \left[\begin{array}{cc} \frac{1}{s+1} & \frac{s+3}{(s+1)(s-2)} \\ \frac{10}{s-2} & \frac{5}{s+3} \end{array}\right], \quad G_2(s) = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 2 & 3 & 1 \\ \hline 1 & 2 & 0 \end{array}\right]$$

$$G_3(s) = \left[\begin{array}{cc|c} -1 & -2 & 1 \\ 1 & 0 & 0 \\ \hline 2 & 3 & 0 \end{array}\right], \quad G_4(s) = \left[\begin{array}{ccc|cc} -1 & -2 & -3 & 1 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 \\ \hline 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2 \end{array}\right]$$