# Robust Control Systems — Mimo Example

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## 1 Problem Statement

The following idealized dynamic model of a distillation column is from Skogestad and Postlethwaite, page 94.

$$G(s) = \frac{1}{75s+1}\begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix}$$

with time in minutes. The plant is ill-conditioned with condition number $\gamma(G) = 141.7$. This plant is difficult to control. The author used this plant to investigate a number of control strategies.

A simple inverse control strategy with integral control may be used here (the inverse control will attempt to cancel the "unwanted" plant poles with controller zeroes)

$$K_I(s) = \frac{k_i}{s}\, G^{-1}(s) = \frac{0.7(1+75s)}{s}\begin{bmatrix} 0.3994 & -0.3149 \\ 0.3943 & -0.3200 \end{bmatrix}$$

We have now $GK_I = K_I G = 0.7/s$. The sensitivity and complementary sensitivity functions

$$S = \frac{s}{s+0.7}\, I \quad T = \frac{1}{1.43s+1}\, I$$

with norms less than unity so the designs should be robust stable.

In Matlab 6+ you would enter the model and controller, using cell arrays, as

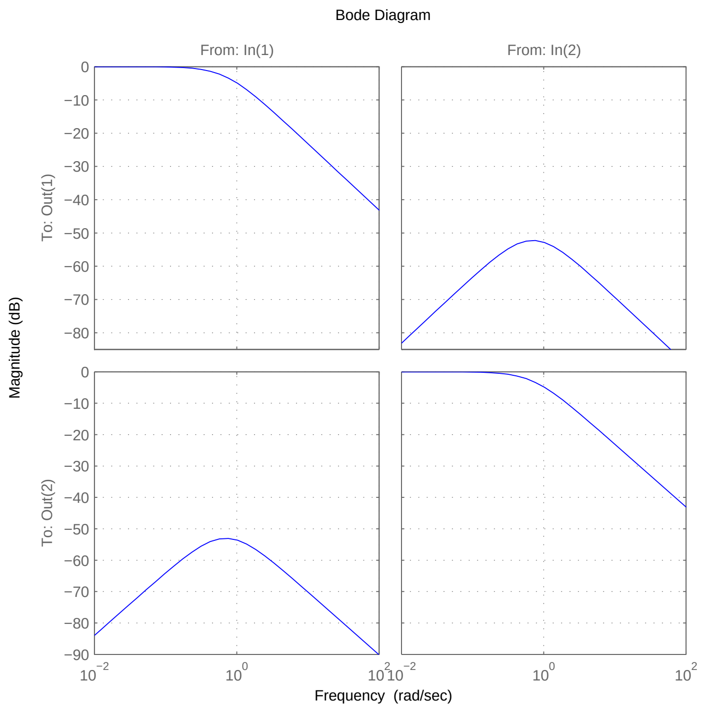
```
nums = {87.8 -86.4;108.2 -109.6};
den  = [75 1];
Gs   = tf(nums,den);

numki = {0.3994*[75 1] -0.3149*[75 1]; 0.3943*[75 1] -0.3200*[75 1]};
```

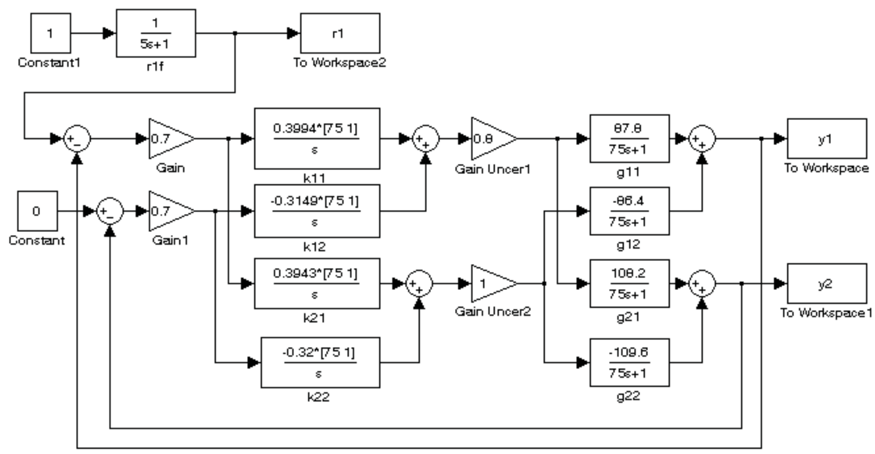
```
denki = [1 0];
Ki    = 0.7 * tf(numki,denki);

Li = Ki*Gs;
sysi = feedback(Li,-eye(2,2));
```

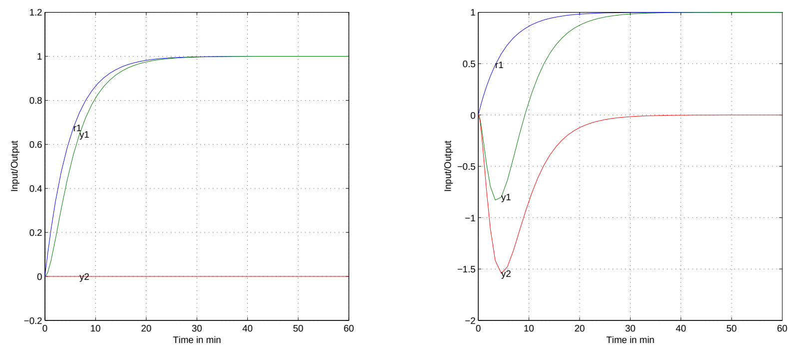
The resultant Bode plot for the system is




The simulation diagram for the system is



with step response (nominal and channel 1 gain perturbed by 20%)




Note that although the feedback system is stable, the system is sensitive to gain changes. This can be confirmed by looking at the singular value decomposition

$$G' = \begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix} = \underbrace{\begin{bmatrix} 0.625 & -0.781 \\ 0.781 & 0.625 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 197.2 & 0.0 \\ 0.0 & 1.39 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0.707 & -0.708 \\ -0.708 & -0.707 \end{bmatrix}}_{V}$$

The gain from the first singular vector $\begin{bmatrix} 0.707 & -0.708 \end{bmatrix}^T$ is 197.2, i.e. the inputs counteract each other. The condition number is $197.2/1.39 = 142$, so the plant is highly ill-conditioned.

## 2 $H_\infty$ Loopshape Controller

In this case we start with the diagonalized plant

$$G_{sd} = \begin{bmatrix} 0.3994 & -0.3149 \\ 0.3943 & -0.3200 \end{bmatrix} \frac{1}{75s+1} \begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix}$$

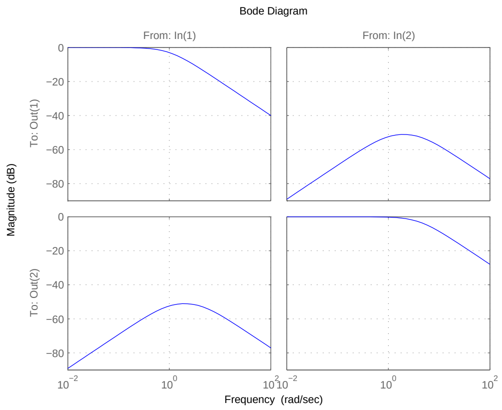
Choose a reference loopshape given by

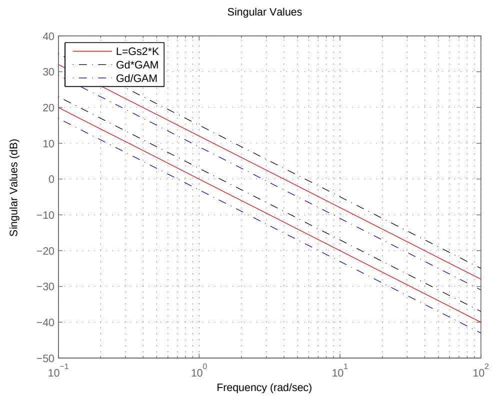
$$G_d = \begin{bmatrix} \frac{1}{s} & 0 \\ 0 & \frac{4}{s} \end{bmatrix}$$

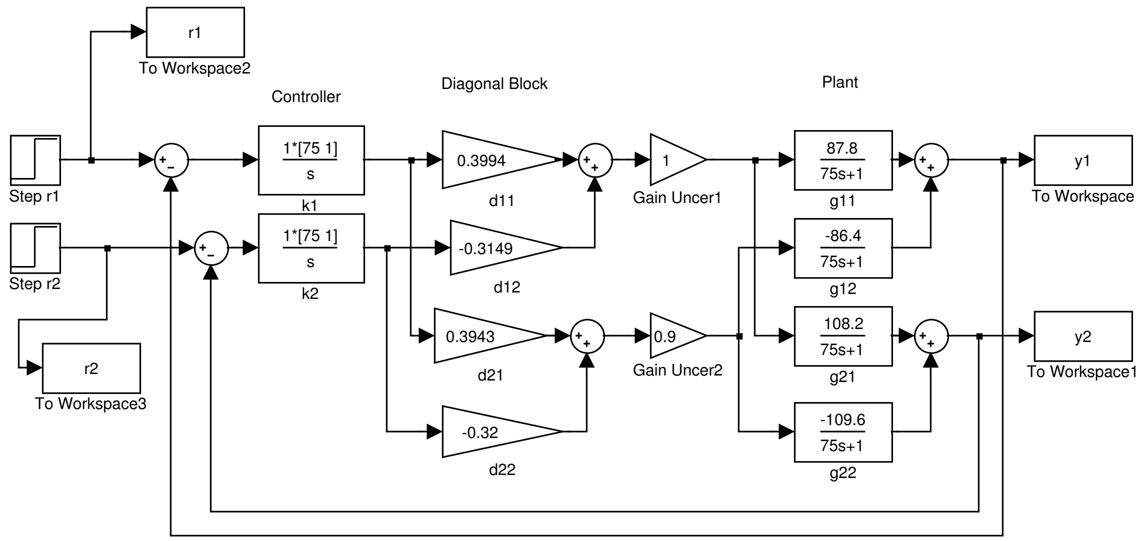
with a controller solution

$$
\begin{aligned}
K_{11} &= \frac{309078.5661s(s+4104)(s+4096)^2(s+0.01333)^3}{s^2(s+4097)^3(s+4104)(s+0.01333)^2} \approx \frac{75.4(s+0.01333)}{s} \\
K_{12} &= \frac{5658.9085s(s+4097)^3(s+0.01333)^3}{s^2(s+4097)^3(s+4104)(s+0.01333)^2} \approx \frac{1.38(s+0.01333)}{s} \\
K_{21} &= \frac{-1675.1228s(s+4103)(s+4096)^2(s+0.01333)^3}{s^2(s+4097)^3(s+4104)(s+0.01333)^2} \approx \frac{-0.41(s+0.01333)}{s} \\
K_{22} &= \frac{1223924.5431s(s+4097)^3(s+0.01333)^3}{s^2(s+4097)^3(s+4104)(s+0.01333)^2} \approx \frac{298.2(s+0.01333)}{s}
\end{aligned}
$$

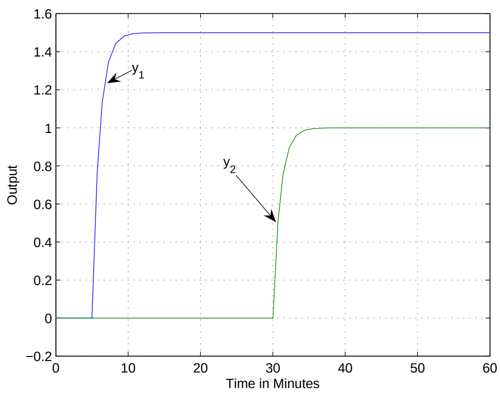
The transfer function Bode plot and SVD frequency match is given below

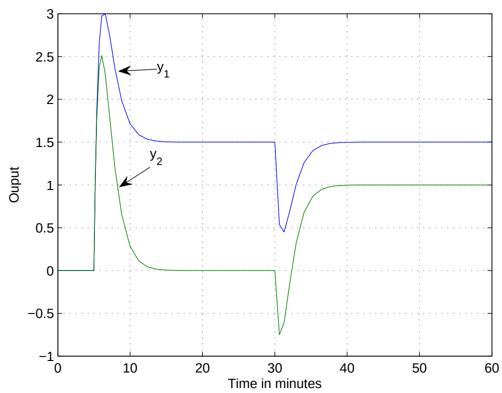





with step response (left diagram with perfect match and right with 10% gain error)




The design is stable, but the decoupling is very sensitive to gain uncertainties (a bad plant stays a bad plant).