Optimal Control: Robustness Enhancement of the LQG Problem by using the LTR Procedure by Doyle and Stein

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JC Doyle and G Stein has presented a process through which the robustness of observer based controllers can be enhanced — see [1] and [2].

1 Robustness Properties of LQR and LQG systems

Consider the following two control loops in Figure 1. In this figure, (a) represents a LQR system with full state feedback and (b) a LQG system with an observer/estimator.

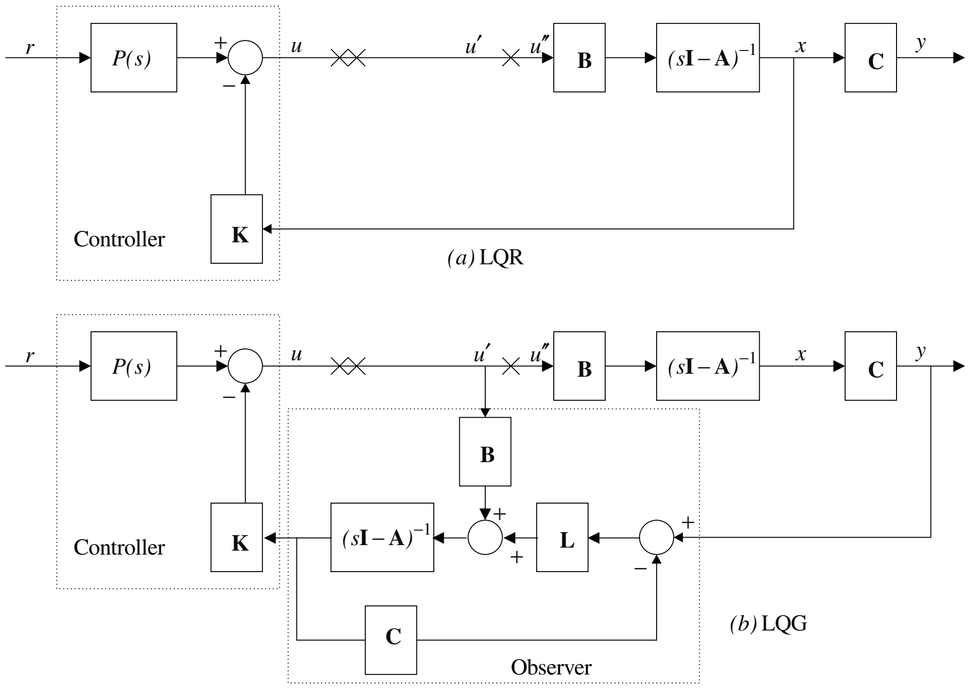

Figure 1: LTI Control Loops: (a) LQR and (b) LQG

The crosses $\times$ and $\times\times$ in Figure 1 indicates points where we will open the control loops to investigate the stability of the system relative to the signals $r, x, \hat{x}, u, u', u''$ and y.

The transfer function properties between the signal breakpoints are:

Property 1 The closed-loop transfer function from $r \to x$

LQR For the LQR system with full state feedback in Figure 1(a)

$$\begin{aligned}
\frac{x}{r}(s) &= \left[(sI - A)\left[I + (sI - A)^{-1}BK\right]\right]^{-1} BP \\
&= \left[(sI - A)(sI - A)^{-1}\left[(sI - A) + BK\right]\right]^{-1} BP \\
&= (sI - A + BK)^{-1}BP
\end{aligned}$$

LQG For the LQG system in Figure 1(b)

$$\begin{aligned}
\hat{x}(s) &= (sI - A + LC)^{-1}\left[Bu + LCx\right] \\
&= (sI - A + LC)^{-1}\left[I + LC(sI - A)^{-1}\right] Bu \\
&= (sI - A + LC)^{-1}(sI - A + LC)(sI - A)^{-1}Bu = x(s) \\
\Rightarrow \frac{x}{r}(s) &= (sI - A + BK)^{-1}BP
\end{aligned}$$

Property 1 is therefore the same for the LQR and LQG systems.

Property 2 The open-loop transfer function from $u' \to u$ with the cross $\times\times$ open

LQR For the LQR system with full state feedback in Figure 1(a) is from inspection

$$\frac{u}{u'}(s) = -K(sI - A)^{-1}B$$

LQG For the LQG system in Figure 1(b) is, using the property $\hat{x} = x$ derived in Property 1

$$\frac{u}{u'}(s) = -K(sI - A)^{-1}B$$

Property 2 is therefore the same for the LQR and LQG systems.

Property 3 The open-loop transfer function from $u'' \to u'$ with the cross $\times$ open

LQR For the LQR system with full state feedback in Figure 1(a) is from inspection

$$\frac{u'}{u''}(s) = -K(sI - A)^{-1}B$$

LQG For the LQG system in Figure 1(b) is, using equations for the observer derived in Property 1

$$\begin{aligned}
u'(s) &= -K(sI - A + LC)^{-1}\left[Bu' + LC(sI - A)^{-1}Bu''\right] \\
&= -K(sI - A + LC + BK)^{-1}LC(sI - A)^{-1}Bu''
\end{aligned}$$

Property 3 is therefore in general not the same for the LQR and LQG cases. To obtain the same transfer function, the observer must have the special property in the LQG case of

$$(sI - A + LC + BK)^{-1}LC = I$$

which is only possible if $LC \rightarrow \infty$. This implies that for any realistic observer bandwidth, the robustness of the LQG system is worse than that of the LQR case (for initial conditions and non-linearities)

Note that equation (1) in [1]

$$L\left[I + C(sI - A)^{-1}L\right] = B\left[C(sI - A)^{-1}B\right]^{-1}$$

simplifies to

$$C(sI - A)^{-1} = L^{-1} + C(sI - A)^{-1}$$

which also means that $L \rightarrow \infty$.

2 LTR (Loop Transfer Recovery)

2.1 Basis

The procedure is based on the adjustment of the observer gains. Suppose the observer gains are parameterised as a function $L(q)$ of a scalar variable q such that

$$\lim_{q\rightarrow\infty} \frac{L(q)}{q} = BW$$

with W a non-singular matrix. The resulting error dynamics of the observer will have poles given by the roots of

$$\psi(s) = \det(sI - A)$$

2.2 LTR Procedure

The procedure is basically a three-step process:

1. Append additional dummy columns to B and zero rows to L to make $C(sI - A)^{-1}B$ and $K(sI - A)^{-1}B$ square ($r \times r$). Ensure that $C(sI - A)^{-1}B$ is minimum phase.
2. Design the Kalman filter with modified noise intensity matrices

   $$Q_q = \mathcal{E}(ww^T) = (Q + qBB^T)\delta(t - \tau)$$

   The observer gains L as $q \rightarrow \infty$ becomes

   $$\lim_{q\rightarrow\infty} \frac{L(q)}{q} \rightarrow BWN^{-1/2}$$

3. Vary the parameter q to obtain sufficient loop recovery.

2.3 LTR Example

Consider the following plant model

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 30 \\ 30 \end{bmatrix} w \\ y &= \begin{bmatrix} 2 & 1 \end{bmatrix} x + v\end{aligned}$$

with $\mathcal{E}(w) = \mathcal{E}(v) = 0; \mathcal{E}(ww^T) = \mathcal{E}(vv^T) = \delta(t-\tau)$. Assume the controller is given by

$$u = -\begin{bmatrix} 50 & 10 \end{bmatrix} x + 50r$$

which places the closed-loop regulator poles at

$$s = -7.0 \pm j2.0$$

Full-state LQR The resulting gain margin is infinite, and the phase margin is 104° at $\omega = 49\,\text{rad/s}$.

LQG The resulting observer closed loop poles are also at $s = -7.0 \pm j2.0$ with the observer feedback matrix $L = [30 \;\; -50]^T$. The transfer function of the controller from y to u'' is now

$$\frac{u''}{y}(s) = \frac{1000(s+2.6)}{(s+42.7)(s-18.7)}$$

which already identifies a problem. The root-locus for the completed system is shown in Figure 2.

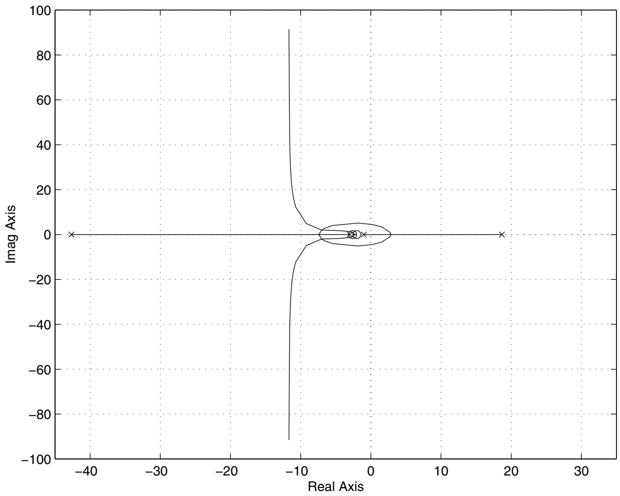

Figure 2: Root-locus for optimal LQG system

Although the poles for the nominal controller and regulator are situated at $s = -7.0 \pm j2.0$, the design is definately not robust for gain changes. Looking at the Nichols chart in in Figure 3, and subtracting 360°, we can see that the phase margin is only 15°.

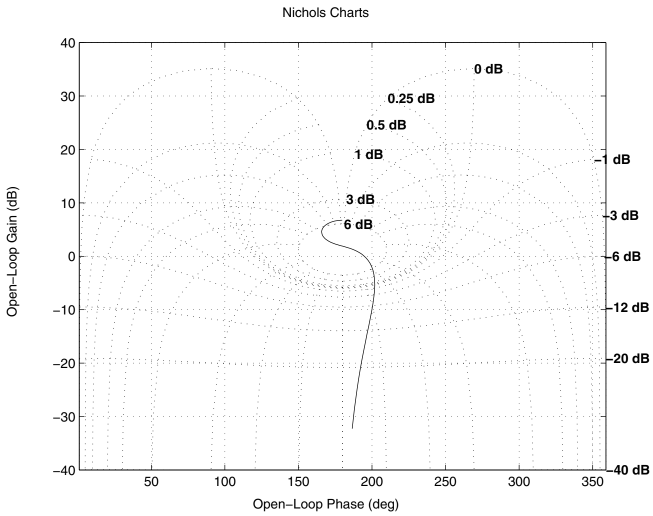

Figure 3: Nichols chart for optimal LQG system

LQG/LTR The new process variance is

$$Q(q) = \begin{bmatrix} 35 \\ -61 \end{bmatrix} \begin{bmatrix} 35 & -61 \end{bmatrix} + q^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 1225 & -2135 \\ -2135 & 3721 + q^2 \end{bmatrix}$$

Also

$$K = \begin{bmatrix} 50 & 10 \\ 0 & 0 \end{bmatrix}; \qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

resulting observer closed loop poles are also at $s = -7.0 \pm j2.0$ with the observer feedback matrix $L = [30 \;\; -50]^T$. The transfer function of the controller from y to u'' is now for $q = 100$

$$\frac{u''}{y}(s) = \frac{2133(s + 3.7)}{(s + 127.3)(s - 5.7)}$$

and the phase margin improved to about 50°.

With $q = 1000$

$$\frac{u''}{y}(s) = \frac{11148(s + 4.7)}{(s + 1020)(s + 1)}$$

and the phase margin improved to about 80°. The system is now also robust stable. The stability improvement is shown on the Nichols chart in Figure 4.

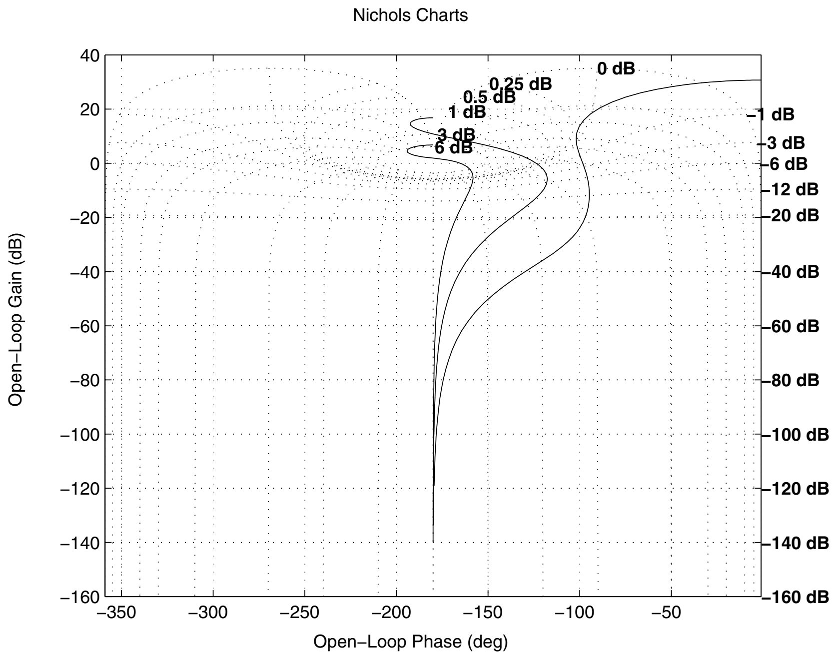

Figure 4: Nichols chart for optimal LQG system, and LTR ratios of $q = 100, 1000$

References

[1] *Robustness with Controllers*, JC Doyle and G Stein, Transactions on Automatic Control, Vol AC-24, pages 607–611

[2] *Multivariable Feedback Design: Concepts for a Classical/Modern Synthesis*, JC Doyle and G Stein, Transactions on Automatic Control, Vol AC-26, pages 4–16