H_∞ Control

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1 Formulation of the H_∞ Problem

1.1 Uncertainty Review

Consider the plant model with feedback as shown in Figure 1.

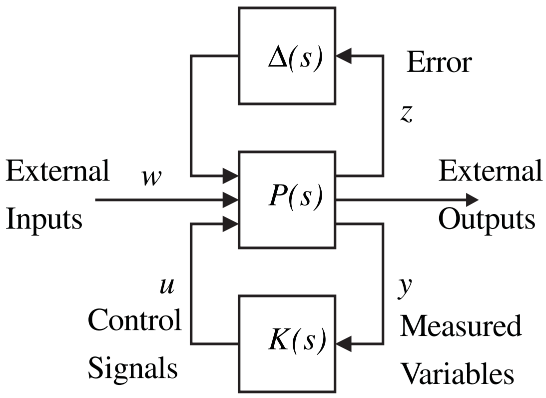


Figure 1: Standard Presentation of Model with Uncertainty

In this case $\Delta(s)$ represents plant parameter uncertainty, $P(s)$ represents the plant nominal transfer function and $K(s)$ represents feedback control.

1.1.1 Sensitivity Reduction

The first mission of the design is to make the system insensitive to external disturbances. This is equivalent to make z as independent of w as possible.

If we ignore the plant perturbation $\Delta(s)$, then the Model simplifies to that shown in Figure 2.

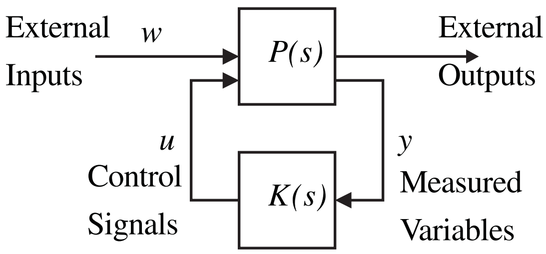


Figure 2: Standard Presentation of Model without Plant perturbation $\Delta(s)$

Suppose $P(s)$ can be partitioned as follows

$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix}$$

so that

$$z = P_{11}w + P_{12}u \qquad y = P_{21}w + P_{22}u$$

then with the feedback law $u = K(s)y$ we can eliminate u and y

$$\begin{aligned} z &= \left[P_{11} + P_{12}K\left(I - P_{22}K\right)^{-1} P_{21}\right] w \qquad (1) \\ &= \mathbf{F}_1(P, K)w \end{aligned}$$

To minimize the error z due to the external inputs w, we must minimize the function $\mathbf{F}_l(P, K)$.

1.1.2 Mixed Performance and Robustness Objective

The following set of characteristics are possible:

- We want to achieve good disturbance rejection from external signals in the low-frequency region. This can be achieved by making the sensitivity $S = (I + PK)^{-1}$ small as $\omega \to 0$.

- Make the closed loop transfer function small at high frequencies limit excitation by noise. This can be achieved by making $T = I - S = I - (I + PK)^{-1}$ small as $\omega \to \infty$.

- Guard against instability from parameter variations. This is achieved by minimizing $K(I + PK)^{-1}$.

We can then formulate the H_∞ problem as the minimization of the function

$$\mathbf{F}_1(P, K) = \begin{bmatrix} W_1 S \\ W_3(I - S) \end{bmatrix}$$

where W_1 and W_3 are frequency-dependent matrices.

2 Solution of the H_∞ Problem

It is possible to formulate the problem in many ways. In the literature a difference is made between the **1-block**, **2-block** and the **4-block** formulations

2.1 Glover-Doyle Algorithm

2.1.1 Formulation

The Glover-Doyle algorithm is the classic formulation on which the `Matlab Robust Control Toolbox` concentrates. This toolbox solves the basic mixed performance and robustness objective.

This algorithm solves a family of stabilising controllers such that

$$\mathbf{F}_l(P, K) \leq \gamma$$

Our search is to find the lowest value of γ for which the above equation has a solution. One possibility is to start with the **LQG** solution and then to reduce it using a binary search.

The plant equations in state space form is

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{11} w + D_{12} u \\ y &= C_2 x + D_{21} w + D_{22} u \end{aligned}$$

and can be represented in the packed matrix form

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

The following assumptions must be satisfied to ensure a solution:

- The pair (A, B_2) must be stabilizable and the pair (C_2, A) detectable.
- With the dimensions of $\dim x = n$, $\dim w = m_1$, $\dim u = m_2$, $\dim z = p_1$ and $\dim y = p_2$, then the $\text{Rank}\, D_{12} = m_2$ and $\text{Rank}\, D_{21} = p_2$ to ensure that they controllers are proper and the transfer function from w to y is non-zero at high frequencies (i.e. all-pass).
- Rank $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix} = n + m_2$ for all frequencies.
- Rank $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix} = n + p_2$ for all frequencies.
- $D_{11} = 0$ and $D_{22} = 0$ will simplify the equations and implies that the transfer functions from u to y and from w to z rolls off at high frequency.

So our simplified problem is

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]$$

2.1.2 Solution

The solution of this problem requires the solving of two Ricatti equations, one for the controller and one for the observer.

The control law is given by

$$u = -K_c \hat{x}$$

and the state estimator equation by

$$\dot{\hat{x}} = Ax + B_2 u + B_1 \hat{w} + Z_\infty K_e (y - \hat{y})$$

where

$$\begin{aligned} \hat{w} &= \gamma^{-2} B_1^T X_\infty \hat{x} \\ \hat{y} &= C_2 \hat{x} + \gamma^{-2} D_{21} B_1^T X_\infty \hat{x} \end{aligned}$$

The controller gain is K_c as for the **LQG** case, and the estimator gain is $Z_\infty K_e$ instead of K_e as for the **LQG** case, with

$$\begin{aligned} K_c &= \tilde{D}_{12}(B_2^T X_\infty + D_{12}^T C_1), \quad \tilde{D}_{12} = (D_{12}^T D_{12})^{-1} \\ K_e &= (Y_\infty C_2^T + B_1 D_{21}^T)\tilde{D}_{21}, \quad \tilde{D}_{21} = (D_{21} D_{21}^T)^{-1} \end{aligned}$$

and

$$Z_\infty = (I - \gamma^{-2} Y_\infty X_\infty)^{-1}$$

The terms X_∞ and Y_∞ are solutions to the controller and estimator Ricatti equations

$$\begin{aligned} X_\infty &= \text{Ric}\begin{bmatrix} A - B_2\tilde{D}_{12}D_{12}^T C_1 & -\gamma^{-2}B_1B_1^T - B_2\tilde{D}_{12}B_2^T \\ -\tilde{C}_1^T\tilde{C}_1 & -(A - B_2\tilde{D}_{12}D_{12}^T C_1 \end{bmatrix} \\ Y_\infty &= \text{Ric}\begin{bmatrix} (A - B_1D_{21}^T\tilde{D}_{21}C_2)^T & -\gamma^{-2}C_1^T C_1 - C_2^T\tilde{D}_{21}C_2 \\ -\tilde{B}_1\tilde{B}_1^T & -(A - B_1D_{21}^T\tilde{D}_{21}C_2 \end{bmatrix} \end{aligned}$$

with $\tilde{B}_1 = B_1(I - D_{21}\tilde{D}_{21}D_{21}^T)$ and $\tilde{C}_1 = B_1(I - D_{12}\tilde{D}_{12}D_{12}^T)$.

We do not carry out these calculations by hand — the tools supplied by the `Matlab Robust Control Toolbox` does just that.

3 Properties of H_∞ Controllers

The following important properties for H_∞ controllers exist:

- The stabilising feedback law $u_2(s) = K(s)y_2(s)$ minimizes the norm of the closed loop transfer function

  $$T_{yu} = G_{11}(s) + G_{12}(s)[I - K(s)G_{22}(s)]^{-1}K(s)G_{21}(s)$$

  The problems we can solve is

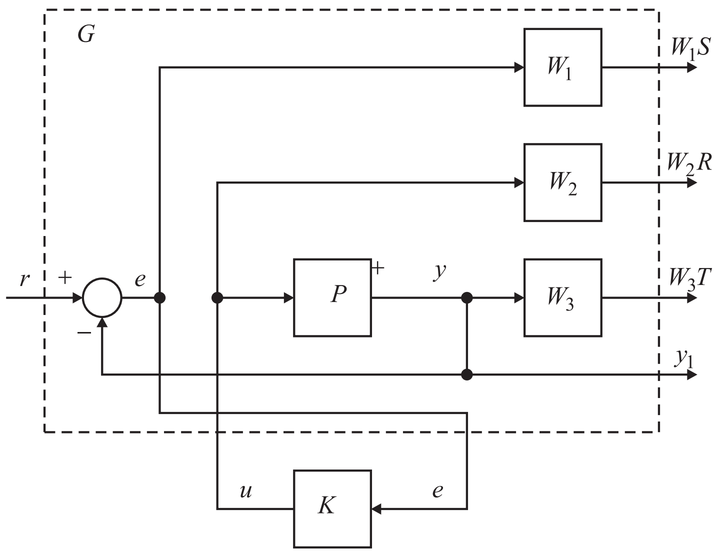
  - Optimal H_2 control: $\min||T_{yu}||_2$
  - Optimal H_∞ control: $\min||T_{yu}||_\infty$
  - Standard H_∞ control: $\min(||T_{yu}||_\infty \leq 1)$

- The H_∞ cost function T_{yu} is all-pass i.e. $\overline{\sigma}(T_{yu}) = 1$ for all values of ω.

- The H_∞ optimal controller (use `hinfopt.m` in `Matlab`) for an n-state augmented plant have at most $n-1$ states.

- The H_∞ sub-optimal controller (use `hinf.m` or the newer `hinfsyn.m` in `Matlab`) for an n-state augmented plant have exactly n states.
- In the weighted mixed sensitivity problem formulation, the H_∞ controller always cancels the stable poles of the plant with its transmission zeroes.
- In the weighted mixed sensitivity problem formulation, the unstable poles of the plant inside the specified bandwidth will be shifted to its mirror image once a H_∞ or H_2 feedback loop is closed.

The implications are that this technique allows very precise frequency-domain loop shaping via suitable weighting strategies. If you augment the plant with frequency dependent weights W_1 to W_3, then the `Matlab` script `hinf` or newer `hinfsyn` or `mixsyn` will find a controller that "shapes" the signals to the inverse of these weights, if it exists. The Matlab function `augw.m` forms the augmented plant



$$G(s) = \left[\begin{array}{c|c} W_1 & -W_1\,P \\ 0 & W_2 \\ 0 & W_3\,P \\ \hline I & -P \end{array}\right]$$

4 Examples

4.1 Example 1

Consider the case of the double integrator

$$G(s) = \frac{1}{s^2}$$

This plant violates the rules for a solution (poles on imaginary axis). Now we must set up the equations carefully. The equation set, in state space form, with the addition of a "disturbance" term representing uncertainty d, is

$$\begin{aligned} \dot{x}_1 &= d + u \\ \dot{x}_2 &= x_1 \end{aligned}$$

with the regulated output (note the inclusion of the control signal to bound it) given by

$$z = \begin{bmatrix} x_2 \\ u \end{bmatrix}$$

with the measurement equation

$$y = x_2 + n$$

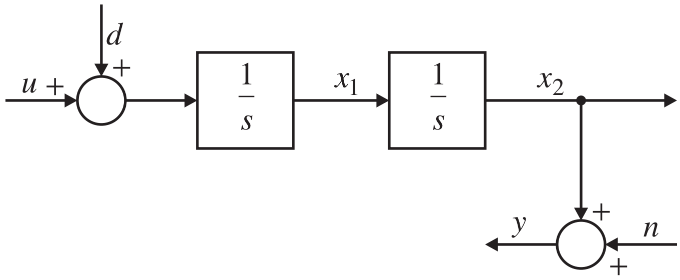
The "noise" term n may include measurement errors or unmodelled high-frequency dynamics — we also need it to ensure the rank condition of D_{21} is met.

Our set of equations are

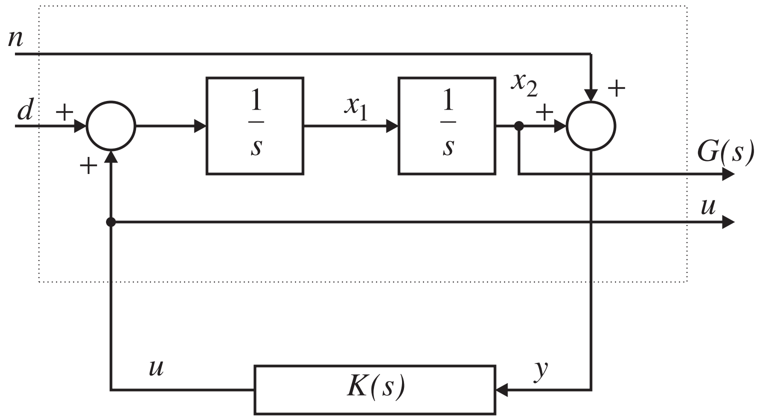
$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$D_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D_{22} = 0$$

In standard format and H_∞ format, the block diagrams of the double integrator is shown in Figure 3.



(a) Standard Format



(b) H_∞ block format

Figure 3: Double integrator example expressed into (a) Standard format and (b) H_∞ format

Collecting the equations in packed matrix form

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] = \left[\begin{array}{cc|cccc} 0 & 0 & 1 & 0 & \vdots & 1 \\ 1 & 0 & 0 & 0 & \vdots & 0 \\ \hline 0 & 1 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & \vdots & 1 \\ \cdots & \cdots & \cdots & \cdots & & \cdots \\ 0 & 1 & 0 & 1 & \vdots & 0 \end{array}\right]$$

The solution to this problem by computer (pre-shifting the poles at the origin and post-shifting the controller poles back) gives

$$\gamma = 2.62, \quad K_c = \begin{bmatrix} 1.59 & 1.08 \end{bmatrix}, \quad K_e = \begin{bmatrix} 1.08 \\ 1.59 \end{bmatrix}, \quad K(s) = \frac{-578.3(s+0.39)}{(s+2.33)(s+220.7)}$$

with the closed loop poles at $\{-0.71, -0.81 \pm j0.91, -220.7\}$

4.2 Example 2

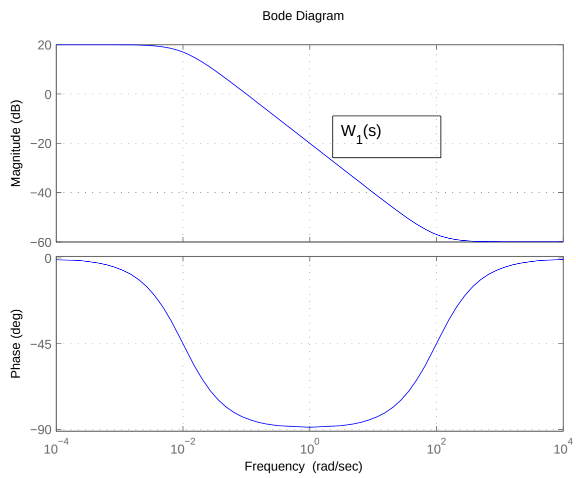
Consider the following plant

$$P(s) = \frac{s-1}{s+1}$$

This need quite agressive control to stabilize. Lets focus on the sensitivity S and choose a weight

$$W_1 = \frac{0.1\,(s+100)}{100\,s+1}$$

with bode plot (remember as $W_1 S \approx 1$, therefore S will track W_1^{-1})



Choose a moderate weight $W_2 = 0.1$ and augment the plant with

```
s  = zpk('s'); P = (s - 1)/(s + 1); % Plant is all-pass with zero in RHP
W1 = 0.1*(s + 100)/(100 * s + 1); % Control S
W2 = 0.1; % Moderate control on u
W3 = []; % Ignore T
G  = augw(P, W1, W2, W3); % Augment the plant
```

The H_∞-controller can be found using

```
[K,CL,GAM] = hinfsyn(G); % or [K,CL,GAM] = mixsyn(G,W1,W2,[]);
```

giving the controller, closed loop and $\gamma = 0.1844$ as

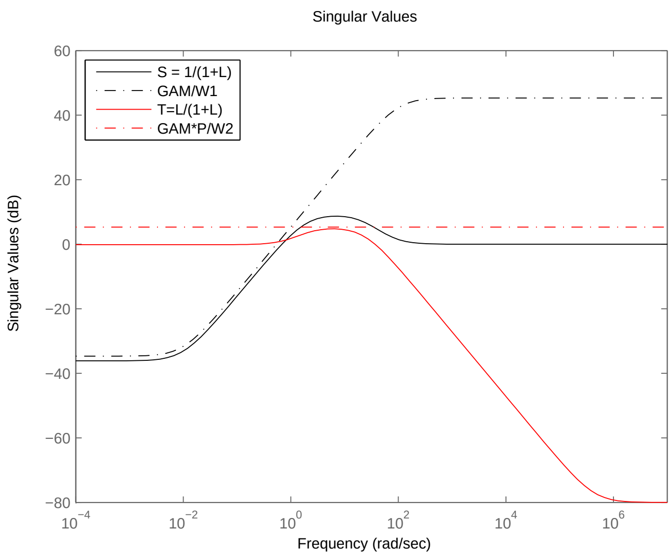
$$
\begin{aligned}
K &= 0.0001 \, \frac{(s+1)(s-438900)}{(s+0.01)(s+69.53)} \\
T_{W_1 S} &= \frac{0.0009999\,(s+100)(s+69.53)(s+1)(s+0.01)}{(s+0.01)(s+1)(s+1.876)(s+23.77)} \\
T_{W_2 R} &= \frac{0.000009999(s+1)^2(s-438900)}{(s+1)(s+1.876)(s+23.77)}
\end{aligned}
$$

How well did the H_∞ controller achieved the objectives? Generate the singular values using

```
L = K*P;  % Form loopgain
S = inv(1+L); % Form S
T = 1-S; % and T

sigma(S,'k',GAM/W1,'k-.',T,'r',GAM*P/W2,'r-.')
legend('S = 1/(1+L)','GAM/W1', 'T=L/(1+L)','GAM*P/W2',2)
```

gives the following singular value plot



4.3 Example 3

Consider the following plant

$$P(s) = \frac{1}{(s+1)(s+2)}$$

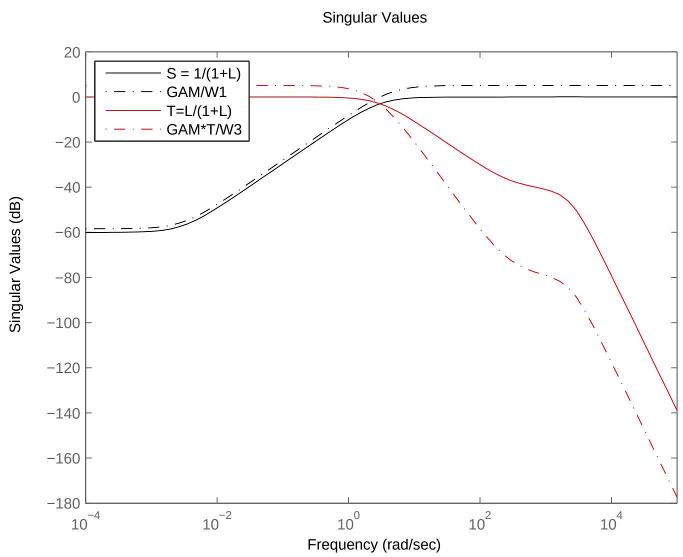
Now choose weights to make the bandwidth about 3 rad/s and the sensitivity S as low as -40 dB at low frequencies. At the same time make the transmission T capable of robustly tolerating uncertainties of about 20 dB. Suitable weights would be (choose $M_s = M_t = 1.5$)

$$\begin{aligned} W_1(s) &= \frac{s/M_s + \omega_s}{s + \omega_s \epsilon_s} = \frac{0.67(s+4.5)}{s+0.003} \\ W_3(s) &= \frac{s + \omega_t/M_t}{\epsilon_t s + \omega_t} = \frac{100(s+2)}{s+300} \end{aligned}$$

Using `hinfsyn` we obtain $\gamma = 1.1973$ and the controller

$$K = \frac{110918138.86(s+300)(s+2)(s+1)}{(s+0.003)(s+1701)(s^2+3636s+6495000)} \approx \frac{3.01(s+2)(s+1)}{s+0.003}$$

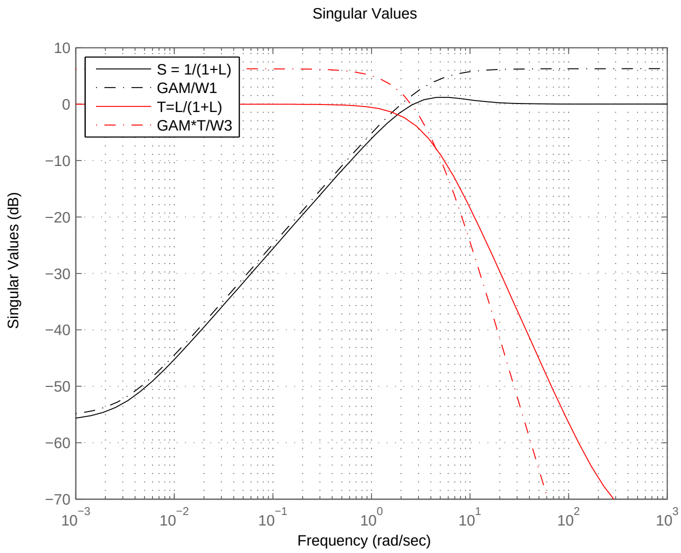
with matching



The controller operates as a lead-type of controller, cancels the plant poles and uses the pole in W_1 as its new controller pole. The match in S is very good but the first-order pole makes the match in T quite bad.

By stiffening the frequency requirement on T by using $W_3' = (W_3)^2$ as weight, reducing the overshoot to $M = 1.2$, we will arrive at $\gamma = 1.72$ and

$$K \approx \frac{13.51(s+2)(s+1)}{(s+0.003)(s+7.09)}$$



The controller can now tolerate 20 dB uncertainty from 10 rad/s and also provide almost -60 dB sensitivity at low frequencies.