1 Formulation of the $H_\infty$ Problem

1.1 Uncertainty Review

Consider the plant model with feedback as shown in Figure 1.

In this case $\Delta(s)$ represents plant parameter uncertainty, $P(s)$ represents the plant nominal transfer function and $K(s)$ represents feedback control.

1.1.1 Sensitivity Reduction

The first mission of the design is to make the system insensitive to external disturbances. This is equivalent to make $z$ as independent of $w$ as possible.
If we ignore the plant perturbation $\Delta(s)$, then the Model simplifies to that shown in Figure 2.

Figure 2: Standard Presentation of Model without Plant perturbation $\Delta(s)$

Suppose $P(s)$ can be partitioned as follows

$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix}$$

so that

$$z = P_{11}w + P_{12}u \quad y = P_{21}w + P_{22}u$$

then with the feedback law $u = K(s)y$ we can eliminate $u$ and $y$

$$z = \begin{bmatrix} P_{11} + P_{12}K(I - P_{22}K)^{-1} P_{21} \end{bmatrix}w$$

$$= F_1(P, K)w \quad (1)$$

To minimize the error $z$ due to the external inputs $w$, we must minimize the function $F_1(P, K)$.

1.1.2 Mixed Performance and Robustness Objective

The following set of characteristics are possible:

- We want to achieve good disturbance rejection from external signals in the low-frequency region. This can be achieved by making the sensitivity $S = (I + PK)^{-1}$ small as $\omega \to 0$.

- Make the closed loop transfer function small at high frequencies limit excitation by noise. This can be achieved by making $T = I - S = I - (I + PK)^{-1}$ small as $\omega \to \infty$. 
• Guard against instability from parameter variations. This is achieved by minimizing $K(I + PK)^{-1}$.

We can then formulate the $H_\infty$ problem as the minimization of the function

$$F_1(P, K) = \begin{bmatrix} W_1 S \\ W_3(I - S) \end{bmatrix}$$

where $W_1$ and $W_3$ are frequency-dependent matrices.

## 2 Solution of the $H_\infty$ Problem

It is possible to formulate the problem in many ways. In the literature a difference is made between the 1-block, 2-block and the 4-block formulations.

### 2.1 Glover-Doyle Algorithm

#### 2.1.1 Formulation

The Glover-Doyle algorithm is the classic formulation on which the Matlab Robust Control Toolbox concentrates. This toolbox solves the basic mixed performance and robustness objective.

This algorithm solves a family of stabilising controllers such that

$$F_1(P, K) \leq \gamma$$

Our search is to find the lowest value of $\gamma$ for which the above equation has a solution. One possibility is to start with the LQG solution and then to reduce it using a binary search.

The plant equations in state space form is

\[
\begin{align*}
\dot{x} &= Ax + B_1w + B_2u \\
z &= C_1x + D_{11}w + D_{12}u \\
y &= C_2x + D_{21}w + D_{22}u
\end{align*}
\]

and can be represented in the packed matrix form

$$G(s) = \begin{bmatrix} A & B_1 & B_2 \\
C_1 & D_{11} & D_{12} \\
C_2 & D_{21} & D_{22} \end{bmatrix}$$
The following assumptions must be satisfied to ensure a solution:

- The pair \((A, B_2)\) must be stabilizable and the pair \((C_2, A)\) detectable.
- With the dimensions of \(\dim x = n, \dim w = m_1, \dim u = m_2, \dim z = p_1\) and \(\dim y = p_2\), then the \(\text{Rank } D_{12} = m_2\) and \(\text{Rank } D_{21} = p_2\) to ensure that they controllers are proper and the transfer function from \(w\) to \(y\) is non-zero at high frequencies (i.e. all-pass).
- \(\text{Rank } \begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix} = n + m_2\) for all frequencies.
- \(\text{Rank } \begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix} = n + p_2\) for all frequencies.
- \(D_{11} = 0\) and \(D_{22} = 0\) will simplify the equations and implies that the transfer functions from \(u\) to \(y\) and from \(w\) to \(z\) rolls off at high frequency.

So our simplified problem is

\[
G(s) = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix}
\]

### 2.1.2 Solution

The solution of this problem requires the solving of two Ricatti equations, one for the controller and one for the observer.

The control law is given by

\[
u = -K_c \hat{x}
\]

and the state estimator equation by

\[
\dot{\hat{x}} = Ax + B_2u + B_1\hat{w} + Z_\infty K_c(y - \hat{y})
\]

where

\[
\hat{w} = \gamma^{-2}B_1^T X_\infty \hat{x}
\]

\[
\hat{y} = C_2\hat{x} + \gamma^{-2}D_{21}B_1^T X_\infty \hat{x}
\]
The controller gain is $K_c$ as for the LQG case, and the estimator gain is $Z_\infty K_e$ instead of $K_e$ as for the LQG case, with

\[
K_c = \hat{D}_{12}(B_2^T X_\infty + D_{12}^T C_1), \quad \hat{D}_{12} = (D_{12}^T D_{12})^{-1}
\]
\[
K_e = (Y_\infty C_2^T + B_1 D_{21}^T) \hat{D}_{21}, \quad \hat{D}_{21} = (D_{21} D_{21}^T)^{-1}
\]

and

\[
Z_\infty = (I - \gamma^{-2} Y_\infty X_\infty)^{-1}
\]

The terms $X_\infty$ and $Y_\infty$ are solutions to the controller and estimator Ricatti equations

\[
X_\infty = \text{Ric} \left[ \begin{array}{cc} A - B_2 \hat{D}_{12} D_{12}^T C_1 & -\gamma^{-2} B_1 B_1^T - B_2 \hat{D}_{12} B_2^T \\ -\hat{C}_1^T \hat{C}_1 & -(A - B_2 \hat{D}_{12} D_{12}^T C_1) \end{array} \right]
\]
\[
Y_\infty = \text{Ric} \left[ \begin{array}{cc} (A - B_1 D_{21}^T \hat{D}_{21} C_2)^T & -\gamma^{-2} C_1^T C_1 - C_2^T \hat{D}_{21} C_2 \\ -\hat{B}_1 \hat{B}_1^T & -(A - B_1 D_{21}^T \hat{D}_{21} C_2) \end{array} \right]
\]

with $\hat{B}_1 = B_1(I - D_{21} \hat{D}_{21} D_{21}^T)$ and $\hat{C}_1 = B_1(I - D_{12} \hat{D}_{12} D_{12}^T)$.

We do not carry out these calculations by hand — the tools supplied by the Matlab Robust Control Toolbox does just that.

### 3 Properties of $H_\infty$ Controllers

The following important properties for $H_\infty$ controllers exist:

- The stabilising feedback law $u_2(s) = K(s)y_2(s)$ minimizes the norm of the closed loop transfer function

\[
T_{yu} = G_{11}(s) + G_{12}(s)[I - K(s)G_{22}(s)]^{-1}K(s)G_{21}(s)
\]

The problems we can solve is

- Optimal $H_2$ control: $\min ||T_{yu}||_2$
- Optimal $H_\infty$ control: $\min ||T_{yu}||_\infty$
- Standard $H_\infty$ control: $\min (||T_{yu}||_\infty \leq 1)$

- The $H_\infty$ cost function $T_{yu}$ is all-pass i.e. $\sigma(T_{yu}) = 1$ for all values of $\omega$.

- The $H_\infty$ optimal controller (use hinfopt.m in Matlab) for an $n$-state augmented plant have at most $n - 1$ states.
• The $H_\infty$ sub-optimal controller (use hinf.m or the newer hinfsyn.m in Matlab) for an $n$-state augmented plant have exactly $n$ states.

• In the weighted mixed sensitivity problem formulation, the $H_\infty$ controller always cancels the stable poles of the plant with its transmission zeroes.

• In the weighted mixed sensitivity problem formulation, the unstable poles of the plant inside the specified bandwidth will be shifted to its mirror image once a $H_\infty$ or $H_2$ feedback loop is closed.

The implications are that this technique allows very precise frequency-domain loop shaping via suitable weighting strategies. If you augment the plant with frequency dependent weights $W_1$ to $W_3$, then the Matlab script hinf or newer hinfsyn or mixsyn will find a controller that ”shapes” the signals to the inverse of these weights, if it exists. The Matlab function augw.m forms the augmented plant

$$G(s) = \begin{bmatrix} W_1 & -W_1 P \\ 0 & W_2 \\ 0 & W_3 P \\ I & -P \end{bmatrix}$$
4 Examples

4.1 Example 1

Consider the case of the double integrator

\[ G(s) = \frac{1}{s^2} \]

This plant violates the rules for a solution (poles on imaginary axis). Now we must set up the equations carefully. The equation set, in state space form, with the addition of a "disturbance" term representing uncertainty \( d \), is

\[
\dot{x}_1 = d + u \\
\dot{x}_2 = x_1
\]

with the regulated output (note the inclusion of the control signal to bound it) given by

\[
z = \begin{bmatrix} x_2 \\ u \end{bmatrix}
\]

with the measurement equation

\[ y = x_2 + n \]

The "noise" term \( n \) may include measurement errors or unmodelled high-frequency dynamics — we also need it to ensure the rank condition of \( D_{21} \) is met.

Our set of equations are

\[
A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}
\]

\[
D_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D_{22} = 0
\]

In standard format and \( H_\infty \) format, the block diagrams of the double integrator is shown in Figure 3.
Figure 3: Double integrator example expressed into (a) Standard format and (b) $H_\infty$ format

Collecting the equations in packed matrix form

$$G(s) = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & \vdots & 1 \\ 1 & 0 & 0 & 0 & \vdots & 0 \\ 0 & 1 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & \vdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & 1 & \vdots & 0 \end{bmatrix}$$

The solution to this problem by computer (pre-shifting the poles at the origin
and post-shifting the controller poles back) gives

\[ \gamma = 2.62, \quad K_c = \begin{bmatrix} 1.59 & 1.08 \end{bmatrix}, \quad K_e = \begin{bmatrix} 1.08 \\ 1.59 \end{bmatrix}, \quad K(s) = \frac{-578.3(s + 0.39)}{(s + 2.33)(s + 220.7)} \]

with the closed loop poles at \{−0.71, −0.81 ± j0.91, −220.7\}

### 4.2 Example 2

Consider the following plant

\[ P(s) = \frac{s - 1}{s + 1} \]

This need quite aggressive control to stabilize. Let’s focus on the sensitivity \( S \) and choose a weight

\[ W_1 = \frac{0.1 (s + 100)}{100 s + 1} \]

with bode plot (remember as \( W_1 S \approx 1 \), therefore \( S \) will track \( W_1^{-1} \))

Choose a moderate weight \( W_2 = 0.1 \) and augment the plant with

\[
\begin{align*}
\text{s} &= \text{zpk('s'); P} = (\text{s} - 1)/(\text{s} + 1); \quad \% \text{Plant is all-pass with zero in RHP} \\
\text{W1} &= 0.1*(\text{s} + 100)/(100 * \text{s} + 1); \quad \% \text{Control S} \\
\text{W2} &= 0.1; \quad \% \text{Moderate control on u} \\
\text{W3} &= []; \quad \% \text{Ignore T} \\
\text{G} &= \text{augw(P, W1, W2, W3); \% Augment the plant}
\end{align*}
\]
The $H_\infty$-controller can be found using

$$[K,CL,GAM] = \text{hinfsyn}(G); \% \text{ or } [K,CL,GAM] = \text{mixsyn}(G,W_1,W_2,[])$$

giving the controller, closed loop and $\gamma = 0.1844$ as

$$K = \frac{0.0001 (s + 1)(s - 438900)}{(s + 0.01)(s + 69.53)}$$
$$T_{W_1S} = \frac{0.0009999 (s + 100)(s + 69.53)(s + 1)(s + 0.01)}{(s + 0.01)(s + 1)(s + 1.876)(s + 23.77)}$$
$$T_{W_2R} = \frac{0.00009999(s + 1)^2(s - 438900)}{(s + 1)(s + 1.876)(s + 23.77)}$$

How well did the $H_\infty$ controller achieved the objectives? Generate the singular values using

$$L = K*P; \% \text{ Form loopgain}$$
$$S = \text{inv}(1+L); \% \text{ Form S}$$
$$T = 1-S; \% \text{ and T}$$

$$\text{sigma}(S,'k',GAM/W_1,'k--','T','r',GAM*P/W_2,'r--')$$
$$\text{legend}('S = 1/(1+L)','GAM/W_1', 'T=L/(1+L)','GAM*P/W_2',2)$$

gives the following singular value plot
4.3 Example 3

Consider the following plant

\[ P(s) = \frac{1}{(s+1)(s+2)} \]

Now choose weights to make the bandwidth about 3 rad/s and the sensitivity \( S \) as low as -40 dB at low frequencies. At the same time make the transmission \( T \) capable of robustly tolerating uncertainties of about 20 dB. Suitable weights would be (choose \( M_s = M_t = 1.5 \))

\[
W_1(s) = \frac{s/M_s + \omega_s}{s + \omega_s \epsilon_s} = \frac{0.67(s + 4.5)}{s + 0.003}
\]

\[
W_3(s) = \frac{s + \omega_t/M_t}{\epsilon_t s + \omega_t} = \frac{100(s + 2)}{s + 300}
\]

Using \texttt{hinfsyn} we obtain \( \gamma = 1.1973 \) and the controller

\[
K = \frac{110918138.86(s + 300)(s + 2)(s + 1)}{(s + 0.003)(s + 1701)(s^2 + 3636s + 6495000)} \approx \frac{3.01(s + 2)(s + 1)}{s + 0.003}
\]

with matching

The controller operates as a lead-type of controller, cancels the plant poles and uses the pole in \( W_1 \) as its new controller pole. The match in \( S \) is very good but the first-order pole makes the match in \( T \) quite bad.
By stiffening the frequency requirement on $T$ by using $W'_3 = (W_3)^2$ as weight, reducing the overshoot to $M = 1.2$, we will arrive at $\gamma = 1.72$ and

$$K \approx \frac{13.51(s + 2)(s + 1)}{(s + 0.003)(s + 7.09)}$$

The controller can now tolerate 20 dB uncertainty from 10 rad/s and also provide almost -60 dB sensitivity at low frequencies.