Real-Time Tracking Filter Evaluation and Selection for Tactical Applications

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Abstract

Five important tracking filters that are often candidates for implementation in systems that must track maneuvering vehicles are compared in terms of tracking accuracy and computer requirements for tactical applications. A rationale for selecting among these filters, which include a Kalman filter, a simplified Kalman filter, an α-β filter, a Wiener filter, and a two-point extrapolator, is illustrated by two examples taken from the authors' recent experience.

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I. Introduction

This paper presents a comparison of tracking accuracy and computer requirements for five important real-time tracking filters evaluated against tactical targets (piloted vehicles). These five filters were selected for comparison since they are of the type often considered for implementation in tactical systems. The results presented can be of immediate value when preliminary filter selection, for applications similar to those presented herein, must be performed. Several considerations in matching filter performance to system requirements are discussed and illustrated by examples taken from tactical system design efforts in which these ideas have actually been applied.

The five filters considered for tracking vehicles, such as aircraft and ships, are the Kalman filter, a simplified Kalman filter, one version of an α-β filter, a Wiener filter, and a two-point extrapolator.

These filters provide results that span the spectrum of tracking accuracies and computer requirements achievable by fixed-structure real-time filters, and are generally well suited for implementation with track-while-scan and other nearly constant data rate tracking sensors. Other filters, such as least-squares filters, polynomial filters, and varieties of adaptive filters, are frequent candidates for implementation, but are not treated explicitly in this paper since, for the tactical situations under consideration, bounds on the performance and computer requirements for these can be determined from the results presented for the five filters considered.

The gain vectors of the first three filters are assumed to be calculated in real time. This permits, first, adaption of the filters to changing tactical environments, and, second, optimal tracking in the presence of missed data points. Stored-gain versions of these filters can, of course, be implemented at considerable computational savings. However, the last two filters considered (Wiener filter and two-point extrapolator) are both examples of stored-gain filters, and the results obtained with these can be used, along with the results obtained using the first three filters, to bound both tracking accuracy and computer requirements for the stored-gain versions of the Kalman and α-β filters.

Although the great majority of air traffic control and tactical weapons systems currently in operation require accurate tracking of piloted, maneuverable vehicles, only scant literature [1], [2] exists on the subject of comparative filter evaluation, and it is believed that the results presented in this paper can usefully expand the available data base in this important area.

II. Sensor and Vehicle Modelling

The tracking systems under consideration utilize sensors that provide measurements of range and bearing. This selection is intended to reflect that this pair of measurements is most common; however, other output measurements such as range rate (Doppler) and elevation are also often available.

The vehicles to be tracked that are considered in this paper are assumed capable of severe maneuvers and can be modelled by the state equations

$$x(k+1) = \Phi x(k) + Gu(k) \qquad (1)$$

where

$$x(k) = \begin{bmatrix} r(k) \\ \dot{r}(k) \\ \theta(k) \\ \dot{\theta}(k) \end{bmatrix}$$

$$= \begin{bmatrix} \text{range at time } k \\ \text{range rate at time } k \\ \text{bearing at time } k \\ \text{bearing rate at time } k \end{bmatrix} = \text{vehicle state vector}$$

$$u(k) = \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}$$

$$= \begin{bmatrix} \text{change in vehicle range rate between} \\ \quad \text{time } k \text{ and time } k+1 \\ \text{change in vehicle bearing rate between} \\ \quad \text{time } k \text{ and time } k+1 \end{bmatrix}$$

$$\Phi = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix} = \text{state transition matrix}$$

$$G = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

T = sampling period.

The sensor observes the vehicle's range and bearing. Both measurements are corrupted by independent additive measurement noises. The output equation is

$$y(k) = \begin{bmatrix} \text{measured rate at time } k \\ \text{measured bearing at time } k \end{bmatrix} = Hx(k) + \nu(k) \qquad (2)$$

where

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$\nu(k) = \begin{bmatrix} \nu_1(k) \\ \nu_2(k) \end{bmatrix}.$$

Assuming the noises $\nu_1(k)$ and $\nu_2(k)$ are independent, the measurement noise covariance matrix, $R(k)$, satisfies

$$R(k) = E[\nu(k)\nu^T(k)] = \begin{bmatrix} \sigma_R{}^2(k) & 0 \\ 0 & \sigma_\theta{}^2(k) \end{bmatrix}.$$

The model (1) is, in each of the two dimensions (r, θ), a simplified version of the discretized target model developed in [3]. Its suitability, as will be discussed later, was demonstrated by verifying that the theoretical tracking accuracies predicted by the Kalman filter covariance matrix agreed to within 25 percent with the experimental accuracies obtained using Monte Carlo simulation techniques with realistic target trajectories. The assumption of zero elements for G_{11} and G_{32} provides satisfactory results for the majority of applications; however, it is expected that conditions could exist where this assumption might result in degraded performance.

The selection of sensor coordinates (r, θ) rather than Cartesian coordinates (x, y) for the filter implementation has been made because the output matrix H assumes the extremely simple form shown, and the measurement noise covariance matrix becomes diagonal. When Doppler measurements are available, this selection of sensor coordinates becomes extremely advantageous because the Cartesian forms for H and $R(k)$ become complex and time varying and often impose computational penalties for real-time implementation.

III. Statistical Description of Target Maneuver

The input sequence $u(k)$ is additive state (or maneuver) noise that results in the vehicle deviating from a constant velocity trajectory. Although the maneuver history and observation noise are not independent, the covariance $E[u(k)\nu^t(j)]$ is zero. Indeed, the radar cross section of a piloted vehicle changes during a maneuver, causing the radar observation noise to depend on the particular maneuver being exercised by the vehicle. However, the conditional expectation $E[\nu^t(k)/u(k) = u^*]$ is zero, although the conditional covariance $E[\nu(k)\nu^t(k)/u(k)=u^*]$ varies with target maneuver.

The maneuver noise is neither white nor Gaussian. For example, the pilot of an aircraft moving at constant velocity will generally not maneuver unless presented with unfavorable terrain or threatened by either radar detection or attacking vehicles. His maneuver will then often be a turn or an increase or decrease in his forward velocity. A typical maneuver probability density (in one dimension) is shown in Fig. 1. The quantity A is the maximum acceleration which the plane-pilot combination can withstand. Values of the density between no maneuver ($u = 0$) and maximum maneuver ($u = \pm A$) are nonzero because 1) the vehicle may not be accelerating at its maximum rate, 2) the projection of a circular maneuver on any dimension can give values of u from $-A$ to A, and 3) the aircraft exhibits approximately a second-order response to a step acceleration input. Clearly, then, the maneuver density is not Gaussian.

Because the execution of a maneuver generally requires considerable time, the maneuver at one sampling period is correlated with the maneuver at the previous (or the next) sampling period. In fact, if the sampling period is short, the maneuver at one sampling period can be cor-

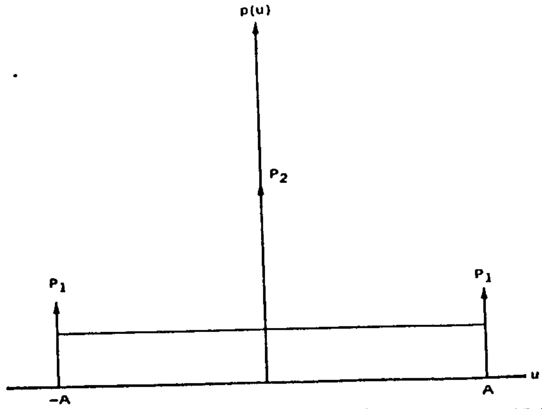

Fig. 1. Typical probability density of target maneuver u.

related with maneuver values separated by many sampling periods. If the maneuver were white, no correlation of maneuver values would exist between different sampling periods.

It is often desirable to whiten the maneuver noise so that system equations to which optimal filtering theory applies can be obtained. This is done, as in [3], by augmenting the state vector to include the maneuver variables u_1 and u_2 and by expressing these variables recursively in terms of white noise sequences. The whitening procedure for a discrete signal is analogous to the procedure developed by Wiener and Kolmogorov to whiten continuous signals [4]. In that case, the power spectral density is factored into symmetrical realizable and unrealizable parts, and the inverse of the realizable part is taken as the transform of the whitening filter.

Using the results obtained in the Appendix, the following augmented set of system equations are obtained that have the white noise sequences $w_1(k)$ and $w_2(k)$ as their only inputs:

$$x_a(k+1) = \Phi_a x_a(k) + G_a w(k) \tag{3a}$$

$$y(k) = H_a x_a(k) + v(k) \tag{3b}$$

where

$$x_a^T(k) = [r(k)\ \dot{r}(k)\ u_1(k)\ \theta(k)\ \dot{\theta}(k)\ u_2(k)] \tag{3c}$$

$$w^T(k) = [w_1(k)\ w_2(k)] \tag{3d}$$

$$\Phi_a = \begin{bmatrix} 1 & T & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & T & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \rho \end{bmatrix} \qquad G_a = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{3e}$$

$$H_a = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{3f}$$

and where

$$Q_a(k) = E[w(k)w^T(k)] = \begin{bmatrix} \sigma_{M_1}^2(1-\rho^2) & 0 \\ 0 & \sigma_{M_2}^2(1-\rho^2) \end{bmatrix}. \tag{3g}$$

The values for the maneuver variances $\sigma_{M_i}^2$ and the correlation coefficient ρ depend on the maneuver characteristics of the vehicles being tracked. As discussed earlier, and as indicated in Fig. 1, these vehicles are assumed to be able to undergo, in each of two orthogonal directions, a maximum acceleration A. The vehicle has a probability P_1 of accelerating at this maximum level (either plus or minus), a probability P_2 of not accelerating at all, and an assumed uniform probability distribution of amplitude $(1-(2P_1+P_2))/2A$ of accelerating between $-A$ and $+A$. The acceleration variable, therefore, has mean zero and variance $(A^2/3)\ (1+4P_1-P_2)$. Consequently, the variables u_1 and u_2, which are assumed independent, have zero means and u_1 has variance

$$\sigma_{M_1}^2 = \frac{A^2T^2}{3}\ (1+4P_1-P_2),$$

while u_2 has variance

$$\sigma_{M_2}^2 = \frac{A^2T^2}{3R^2}\ (1+4P_1-P_2)$$

where R is the vehicle range from the sensor and where all these quantities have appropriate units.

The dynamic equations obtained for the target model, although simple, provide a suitable representation of the target motion under consideration. This has been verified by demonstrating, using Monte Carlo simulation techniques, that the theoretical filter covariance elements and the statistics of the empirical tracking errors resulting from using these filters against a large number of realistic target trajectories (including radial, crossing, and maneuvering paths being traversed at each of three speeds and two ranges by a variety of target vehicles) agreed to within 25 percent. Since a good model combines simplicity with reliable representation of the modelled phenomena, the linear target model developed here must be considered effective. It should be noted that in the case of aircraft targets, it is the resulting aircraft motion in the atmosphere, not the response of the craft to pilot and environmental inputs, that is being modelled. The equations for the latter are much more complex, highly nonlinear, and do not characterize the important phenomena required for the tracking function. Similar remarks can be made for surface and subsurface vehicles.

IV. Filter Description

Five types of filters are considered as potential candidate algorithms for tracking vehicles that are described by the model just discussed. These filters are the Kalman filter, a simplified Kalman filter, a Wiener filter, an α-β filter, and a two-point extrapolator.

A. The Kalman Filter

The Kalman filter uses the augmented version of the model presented earlier in order to obtain white excitation (maneuver) noise, a requirement for the filter's optimality. Of all the filters being examined, the Kalman filter is the most sophisticated, the most accurate, and the most costly to implement. The filter equations are [6]:

$$\hat{x}_a\left(\frac{k+1}{k}\right) = \Phi_a \hat{x}_a\left(\frac{k}{k}\right) \tag{4a}$$

$$\hat{x}_a\left(\frac{k}{k}\right) = \hat{x}_a\left(\frac{k}{k-1}\right) + P_a\left(\frac{k}{k-1}\right) \cdot H_a^T \left[H_a P_a\left(\frac{k}{k-1}\right) H_a^T + R(k)\right]^{-1} \left[y(k) - H_a \hat{x}_a\left(\frac{k}{k-1}\right)\right] \tag{4b}$$

where

$$P_a\left(\frac{k}{k-1}\right) = \Phi_a P_a\left(\frac{k-1}{k-1}\right)\Phi_a^T + G_a Q_a(k-1) G_a^T \tag{4c}$$

$$P_a\left(\frac{k}{k}\right) = P_a\left(\frac{k}{k-1}\right) - P_a\left(\frac{k}{k-1}\right) \cdot H_a^T \left[H_a P_a\left(\frac{k}{k-1}\right) H_a^T + R(k)\right]^{-1} H_a P_a\left(\frac{k}{k-1}\right). \tag{4d}$$

In these equations,

$\hat{x}_a\left(\frac{k}{k}\right)$ = filtered estimate of $x_a(k)$

$\hat{x}_a\left(\frac{k+1}{k}\right)$ = predicted estimate of $x_a(k+1)$

$P_a\left(\frac{k}{k}\right)$ = covariance of filtered estimate

$P_a\left(\frac{k}{k-1}\right)$ = covariance of predicted estimate.

The filter is initialized on the basis of two observations, as follows:

$$\hat{x}_a\left(\frac{2}{2}\right) = \begin{bmatrix} y_1(2) \\ \frac{1}{T}[y_1(2) - y_1(1)] \\ 0 \\ y_2(2) \\ \frac{1}{T}[y_2(2) - y_2(1)] \\ 0 \end{bmatrix}. \tag{5a}$$

The nonzero elements of the corresponding covariance matrix, $P_a(2/2)$, are

$$\begin{aligned}
P_{11} &= \sigma_R^2 & P_{44} &= \sigma_\theta^2 \\
P_{22} &= \sigma_{M_1}^2 + \left(\frac{2\sigma_R^2}{T^2}\right) & P_{55} &= \sigma_{M_2}^2(1) + \left(\frac{2\sigma_\theta^2}{T^2}\right) \\
P_{33} &= \sigma_{M_1}^2 & P_{66} &= \sigma_{M_2}^2(1) \\
P_{12} = P_{21} &= \left(\frac{\sigma_R^2}{T}\right) & P_{45} = P_{54} &= \left(\frac{\sigma_\theta^2}{T}\right) \\
P_{13} = P_{31} &= 0 & P_{46} = P_{64} &= 0 \\
P_{23} = P_{32} &= \rho\sigma_{M_1}^2 & P_{56} = P_{64} &= \rho\sigma_{M_2}^2(1).
\end{aligned} \tag{5b}$$

The values for σ_R^2 and σ_θ^2, the variances of the single-look sensor measurement errors in range and bearing, must be derived for each sensor tested. The quantity $\sigma_{M_1}^2$ should be calculated as discussed previously, and

$$\sigma_{M_2}^2(1) = \left(\frac{\sigma_{M_1}^2}{y_1^2(1)}\right).$$

B. The Simplified Kalman Filter

By simplifying the maneuver model used in the Kalman filter, the state vector can be reduced from six to four elements, and the number of independent components of the covariance matrix from ten to six. The model simplification is achieved by assuming (incorrectly) that the vehicle's change in velocity is uncorrelated between samples; i.e., the maneuver is white. The regular Kalman filter requires two augmented state variables in order to whiten the target maneuver and place the encounter in the theoretical framework necessary to assure that the filter is optimal. If the maneuver is assumed white, no augmentation need be performed and the simplification just discussed occurs. This simplified Kalman filter can also be referred to, therefore, as an unaugmented Kalman filter. The utility of this filter is greatest, therefore, when either the sensitivity of tracking performance to assumed maneuver correlation is small, or when the target maneuver approaches whiteness relative to the sensor data rate. The equations for this filter are

$$\hat{x}\left(\frac{k+1}{k}\right) = \Phi\hat{x}\left(\frac{k}{k}\right) \tag{6a}$$

$$\hat{x}\left(\frac{k}{k}\right) = \hat{x}\left(\frac{k}{k-1}\right) + P\left(\frac{k}{k-1}\right) \cdot H^T\left[HP\left(\frac{k}{k-1}\right)H^T + R(k)\right]^{-1}\left[y(k) - H\hat{x}\left(\frac{k}{k-1}\right)\right] \tag{6b}$$

where

$$P\left(\frac{k}{k-1}\right) = \Phi P\left(\frac{k-1}{k-1}\right)\Phi^T + GQ(k-1)G^T \tag{6c}$$

$$P\left(\frac{k}{k}\right) = P\left(\frac{k}{k-1}\right) - P\left(\frac{k}{k-1}\right)$$

$$\cdot H^T \left[HP\left(\frac{k}{k-1}\right)H^T + R(k)\right]^{-1} HP\left(\frac{k}{k-1}\right). \quad (6d)$$

In these equations, all quantities except the following have previously been defined:

$\hat{x}\left(\frac{k}{k}\right)$ = filtered estimate of $x(k)$

$\hat{x}\left(\frac{k+1}{k}\right)$ = predicted estimate of $x(k+1)$

$Q(k-1)$ = covariance of "assumed white" maneuver noise

$$= \begin{bmatrix} \sigma_{M_1}^2 & 0 \\ 0 & \dfrac{\sigma_{M_1}^2}{\hat{x}_1^2(k-1/k-1)} \end{bmatrix}$$

$P\left(\frac{k}{k}\right)$ = covariance of filtered estimate

$P\left(\frac{k}{k-1}\right)$ = covariance of predicted estimate.

The simplified Kalman filter is initialized on the basis of two observations, as follows:

$$\hat{x}\left(\frac{2}{2}\right) = \begin{bmatrix} y_1(2) \\ \frac{1}{T}[y_1(2) - y_1(1)] \\ y_2(2) \\ \frac{1}{T}[y_2(2) - y_2(1)] \end{bmatrix}. \quad (7a)$$

The nonzero elements of the corresponding covariance matrix, $P(2/2)$, are

$$P_{11} = \sigma_R^2 \qquad P_{33} = \sigma_\theta^2$$

$$P_{22} = \sigma_{M_1}^2 + \left(\frac{2\sigma_R^2}{T^2}\right) \qquad P_{44} = \sigma_{M_2}^2 + \left(\frac{2\sigma_\theta^2}{T^2}\right)$$

$$P_{12} = P_{21} = \left(\frac{\sigma_R^2}{T}\right) \qquad P_{34} = P_{43} = \left(\frac{\sigma_\theta^2}{T}\right). \quad (7b)$$

C. The α-β Filter

The α-β filter considered in this paper is one of many varieties possible in this class, is more easily implemented than either the Kalman or the simplified Kalman filters, and has been selected for evaluation since it is utilized extensively in tactical applications. Because it is designed to minimize the mean-squared error in filtered position and velocity under the assumption of straight line target motion, it has little capability to track severely maneuvering vehicles. For this reason, various maneuver detection devices are often attached to improve its performance. The filter investigated here has no maneuver detector.

Other α-β filters have been designed that can utilize target maneuver statistics of the type developed earlier. If the performance criterion is dynamic minimization of the total mean-square filtering errors, the filter then takes the form of the Kalman filter. If other performance criteria are utilized, or if the design objectives change (such as, for example, to minimize the tracking errors against a specific trajectory rather than a general class of trajectories), then the corresponding α-β filters take even different forms. An important early paper on one special class of α-β filters is [7]. In the remainder of this paper, the comments on the α-β filter reflect the constraints of the particular filter selected, not α-β filters in general.

The α-β filter examined has no provision to adapt to different target types, as do the Kalman filters, since maneuver statistics are not taken into account. The equations for the α-β filter evaluated are

$$\hat{x}\left(\frac{k+1}{k}\right) = \Phi\hat{x}\left(\frac{k}{k}\right) \quad (8a)$$

$$\hat{x}\left(\frac{k}{k}\right) = \hat{x}\left(\frac{k}{k-1}\right) + K(k)\left[y(k) - H\hat{x}\left(\frac{k}{k-1}\right)\right] \quad (8b)$$

where

$$K^T(k) = \begin{bmatrix} \alpha(k) & \dfrac{\beta(k)}{T} & 0 & 0 \\ 0 & 0 & \alpha(k) & \dfrac{\beta(k)}{T} \end{bmatrix} \quad (9a)$$

and

$$\alpha(k) = \frac{F(k)}{[I(k) + F(k)]} \quad (9b)$$

$$\beta(k) = B''(k)T \quad (9c)$$

$$B''(k) = \frac{B'(k)}{[I(k) + F(k)]} \quad (9d)$$

$$I(k) = \frac{N(k)}{N(k-1)} \quad (9e)$$

$$N(k) = \sigma_R^2(k) + \hat{x}_1^2\left(\frac{k}{k-1}\right)\sigma_\theta^2 \quad (9f)$$

$$F(k) = \alpha(k-1) + [B'(k) + B''(k-1)]T \quad (9g)$$

$$B'(k) = L(k-1)T + B''(k-1) \quad (9h)$$

$$L(k) = L(k-1) - B''^2(k)[1 + F(k)]. \quad (9i)$$

The α-β filter is initialized as follows:

$$\hat{x}\left(\frac{2}{2}\right) = \begin{bmatrix} y_1(2) \\ \frac{1}{T}[y_1(2) - y_1(1)] \\ y_2(2) \\ \frac{1}{T}[y_2(2) - y_2(1)] \end{bmatrix} \quad (10a)$$

$$\alpha(2) = 1$$

$$\beta(2) = 1$$

$$B''(2) = \frac{1}{T}$$

$$F(2) = 0$$

$$L(2) = \frac{2}{T^2}. \qquad (10b)$$

D. The Wiener Filter

The Wiener filter is simpler than any of the three preceding filters. It is a constant-gain filter, in contrast to the previous filters which have gain vectors that change constantly with time. The gain vector used in the Wiener filter is the steady-state gain vector of the regular Kalman filter and is calculated off-line and stored in the computer. When, as in the cases considered here, steady state is reached quickly by the Kalman filter, the Wiener filter performs equivalently to the Kalman filter. (This result was also suggested by a theoretical analysis in [1].) Because it has constant gain, the Wiener filter requires no auxiliary equations to be solved and requires very little computer storage. Because its gain is derived from the Kalman filter, which accounts for target maneuver statistics directly, it is adaptable to a variety of vehicles and can track both maneuvering and nonmaneuvering vehicles well.

The equations for the Wiener filter are

$$\hat{x}_a\left(\frac{k+1}{k}\right) = \Phi_a \hat{x}_a\left(\frac{k}{k}\right) \qquad (11a)$$

$$\hat{x}_a\left(\frac{k}{k}\right) = \hat{x}_a\left(\frac{k}{k-1}\right) + K\left[y(k) - H_a \hat{x}_a\left(\frac{k}{k-1}\right)\right] \qquad (11b)$$

where K is the steady-state gain vector of the corresponding Kalman filter for the given sensor and vehicle class.

The Wiener filter is initialized identically as the Kalman filter, except that no covariance elements are calculated.

An alternative Wiener filter would be the one corresponding to the simplified Kalman filter, rather than the augmented Kalman filter. Since its state vector contains four rather than six elements, such a filter would provide considerable computational advantages over the Wiener filter discussed here. In fact, this "simplified Wiener filter" would generally be preferred when the tracking performances provided by the augmented and simplified Kalman filters are comparable.

E. The Two-Point Extrapolator

The four filters just described are recursive and are examples of filters "with memory." A general block diagram representation of the recursive filters considered is shown in Fig. 2. Only the system models and the way in which the filter gain vector is calculated differ in the various filter formulations.

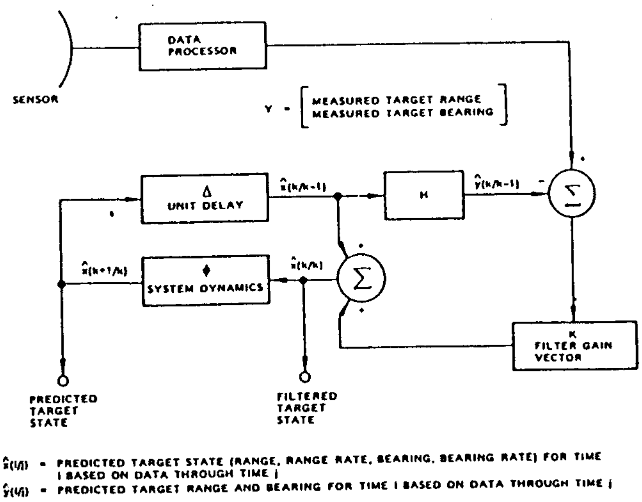

Fig. 2. Block diagram representation of the recursive filters considered.

The simplest "filter" that can be implemented is the "almost memoryless" two-point extrapolator. This filter uses the last data point to determine vehicle range and bearing, and the last two data points to determine target range rate and bearing rate. No filter simpler than the two-point extrapolator can estimate vehicle kinematics. Because this filter has essentially no memory, previous data points do not prejudice predictions, so that the maneuvering and nonmaneuvering vehicles previously described are tracked equally well (badly).

The equations for the two-point extrapolator are

$$\hat{x}\left(\frac{k+1}{k}\right) = \Phi \hat{x}\left(\frac{k}{k}\right) \qquad (12a)$$

$$\hat{x}_1\left(\frac{k}{k}\right) = y_1(k) \qquad (12b)$$

$$\hat{x}_2\left(\frac{k}{k}\right) = \left(\frac{1}{T}\right)[y_1(k) - y_1(k-1)] \qquad (12c)$$

$$\hat{x}_3\left(\frac{k}{k}\right) = y_2(k) \qquad (12d)$$

$$\hat{x}_4\left(\frac{k}{k}\right) = \left(\frac{1}{T}\right)[y_2(k) - y_2(k-1)]. \qquad (12e)$$

V. Comparison of Filter Accuracies

In order to evaluate the five filter algorithms as they would perform in a variety of tactical environments, a spectrum of vehicle types, tracking sensors, and track data entry procedures was selected for analysis. The vehicle types were classified as: 1) 200 to 1000 ft/s air targets and 2) 20 to 50 knot surface and subsurface targets. The sensor types were classified as: 1) air search radars with T = 10 seconds, σ_R=1000 yards, σ_θ = 1 degree; 2) surface and air search radars with T = 1 second, σ_R = 15 yards, σ_θ = 0.1 degree; and 3) radars and sonars with T = 3 seconds, σ_R = 100 yards, σ_θ = 0.4 degree.

Filter Type / Target Type	Air	Air	Air	Surface-Subsurface
Sensor Type	Air Search Radar	Surface and Air Search Radar	Surface and Air Search Radar	Radar, Sonar
Data Entry	Manual	Manual	Automatic	Manual
Two-Point Extrapolator	3	3	3	3
Wiener Filter	0	0	0	0
α-β Filter	2	0	2	1
Simplified Kalman Filter	0	0	0	0
Kalman Filter	0	0	0	0

Key: 0 = Within 20 percent of the Kalman filter
1 = 20-40 percent worse than the Kalman filter
2 = 40-70 percent worse than the Kalman filter
3 = More than 70 percent worse than the Kalman filter

Fig. 3. Synopsis of the accuracy comparison of the five tracking filters.

The procedures for entering data into the tracking computer were classified as: 1) automatic and 2) manual. In the manual mode, the track data is entered into the computer by a man who observes the video on a screen and tags it for entry. In the automatic mode, the man is not in the loop.

Because of manual loading, the operator can only update a limited number of tracks in a given time interval, thereby reducing the effective system data rate. Because the operator introduces inaccuracies by his inexact tagging of the track returns, independent additive errors in range and bearing are included in the single-look measurements. When manual data entry was considered in this analysis, it was assumed that the minimum tag time per target was 3.0 seconds, and that the tag position accuracy was uniformly distributed with maximum error of 0.05 inch on a 5.0 inch display screen. When automatic data entry was considered, the system data rate equalled the sensor data rate and no additional measurement errors were included as a result of the automatic entry.

Maneuver statistics (A, P_1, P_2, λ) were selected to match each vehicle, and sensor statistics (σ_R^2, σ_θ^2, T) were selected for each combination of sensor and data entry scheme evaluated. Three trajectories were constructed for each vehicle—a radial trajectory, in which the vehicle closes directly toward the sensor, a crossing trajectory, in which the vehicle remains at essentially constant range from the sensor, and a maneuvering trajectory, in which the vehicle undergoes severe S-turns, the magnitudes of which were selected to be compatible with the vehicle characteristics.

Twenty-five Monte Carlo trials were made for each combination of tracking filter, vehicle class, vehicle trajectory, tracking sensor, and data entry procedure. Twenty-five cases were deemed sufficient, since only about a 15 percent difference in data was observed when the results of 10 and 25 cases were examined. Experimental filtered and predicted accuracies in vehicle range, bearing, speed, and course were then calculated. Fig. 3 shows a representative summary of the results, in which the steady-state one-sample-ahead prediction accuracies of each filter were compared on a percentage basis to that of the Kalman filter. The entries in the figure were determined by averaging the experimentally obtained percentage degradations in each of the range, bearing, course, and speed coordinates.

The Kalman filter, simplified Kalman filter, and Wiener filter generally performed within 20 percent of each other. This occurred because the transient responses of these filters were short lived relative to the tracking period, and because the correlation coefficient ρ was often small for the data rates and vehicle types evaluated. The α-β filter performed, on the average, about 50 percent worse than the Kalman filter, with the greatest degradation occurring for maneuvering vehicles. Since the α-β filter is similar in many ways to a least-squares filter, its gain vector quickly becomes too small to correct for the large estimation errors resulting from target maneuvers. The two-point extrapolator uniformly performed more than 70 percent worse than the Kalman filter. Since sensor and vehicle characteristics cannot be taken into account, this filter cannot be "tuned" as the others can.

Filter Type	Initialization		Main Loop	
	Time*	Core Locations†	Time*	Core Locations†
Two Point Extrapolator	7	15	7	15
Wiener Filter	8	29	21	33
α-β Filter	40	46	44	58
Simplified Kalman Filter	51	54	81	71
Kalman Filter	54	67	100	100

*Percentage of the computer time required by the Kalman filter in the main loop.
†Percentage of the core locations required by the Kalman filter in the main loop.

Fig. 4. Comparison of the computer requirements for the five tracking filters.

VI. Comparison of Computer Requirements

The computer time and computer storage requirements were determined for the initialization and "main loop" phases of each filter algorithm. Fig. 4 shows the results as normalized to the computer requirements of the Kalman filter. The filter implementation requirements increase in the following order: two-point extrapolator, Wiener filter, α-β filter, simplified Kalman filter, Kalman filter. Moreover, the "complexity factor" between successive filters is about two-to-one.

The numbers appearing in Fig. 4 depend, to a small extent, on the programmer's expertise and, to a larger extent, on the computer hardware being employed. The filter equations used for computer sizing were obtained by simplifying the expressions resulting from expansion into component form of each of the filter algorithms presented in this paper. The basic simplicity of the filter equations limits significantly the differences among the logic statements obtained by different programmers. Similarly, the relative computer storage requirements illustrated in the figure should depend only slightly on the computer hardware being utilized.

The relative computation times were determined for a typical computer utilized in tactical system applications. The times to complete an add, a subtract, and a store were assumed equal; the time to complete a multiply was assumed to be five times that of an add; and the time to complete a divide was assumed to be nine times that of an add. Although these figures are representative, they vary from computer to computer, and some modification of the computation times appearing in Fig. 4 may be desirable when computers with different properties are being considered.

VII. Filter Selection: Two Examples

In most applications, the answer to the question, "Which filter is most accurate?", does not alone determine filter selection. Indeed, the following seven questions must all be answered in the filter selection process to obtain the "best" filter for a particular system:

1) What are the actual filter accuracies of each filter?
2) What are the tracking accuracy requirements of the system?
3) What are the relative filter accuracies?
4) How sensitive is system performance to tracking accuracy?
5) What are the computer requirements of the filters?
6) What are the computer limitations of the system?
7) What auxiliary functions, if any, depend on knowledge of tracking accuracies?

The list shows that filter selection involves careful balancing of filter accuracies, filter implementation requirements, and system performance goals and limitations. Too often the system implications, which frequently dominate other considerations, are ignored in filter selection.

Answers to the first two questions eliminate filters whose accuracies are not sufficient to permit satisfactory system performance. Answers to the fifth and sixth eliminate filters whose computer requirements are too severe to allow efficient system operation. Answers to the third and fourth questions show how critical a role tracking accuracy plays in determining system performance, and indicates how close to the system accuracy limits a filter can be before attention should be directed to the effects of errors in modelling and incorrect threat parameter selection. The answer to question seven indicates to what degree accurate measures of tracking accuracy are required and, consequently, whether or not, and what type of auxiliary equations will have to be included in filter implementation.

The ideas presented herein have actually been applied during tactical system design efforts. In two of these efforts, the authors were charged with the tasks of tracking filter selection and mechanization to meet overall system performance objectives within specified constraints. The two examples to follow, which illustrate the role of system requirements in filter selection, are taken from these system design efforts.

	Bearing (deg)	Range (yd)	Time/Cycle (μs)	Functions
System Requirements	2.0	500	1000	None
Two Point Extrapolator	1.9	490	50	
Wiener Filter	1.25	350	150	
α-β Filter	1.4	380	325	
Simplified Kalman Filter	1.2	350	600	
Kalman Filter	1.1	320	740	
Sensitivity of system performance to tracking accuracy = 0.25.				

Fig. 5. Filter and system parameters for the first example.

For the first example, typical system requirements and filter performance parameters are shown in Fig. 5. It is seen that all the filters considered provide sufficient accuracy and computational ease to meet all system constraints. However, increasing tracking errors by 20 percent would cause 5 percent degradation in system performance, which would be unacceptable in this application. Since the tracking error statistics illustrated in the figure depend on assumed threat and environment parameters that may not be realized precisely in the tactical situation, these statistics can often be in error by as much as 20 percent. When such conditions exist, the two-point extrapolator provides unsatisfactory system performance, and, consequently, was eliminated as a suitable candidate for implementation. The Wiener filter was then selected for implementation since, first, it provided accuracy well within the minimum system requirements and only 15 percent below that of the Kalman filter and, second, it achieved this accuracy with one-fifth the computational requirements of the Kalman filter. Selection of the Wiener filter, therefore, was made because it best combined accuracy and computational ease and thereby permitted possible system computational growth.

In the second example, a tracking filter was required to provide accurate tracking consistent with rather severe computer time and storage constraints. In addition, estimates of actual tracking accuracies are required to satisfactorily perform other system functions, which in this case included weapon kill probability calculations. Fig. 6 presents the system and filter parameters corresponding to this example. It was found that, except for the two-point extrapolator, the tracking accuracy requirements were essentially satisfied by all the filters considered. The α-β filter, however, provided unsatisfactory performance when the tracked vehicles executed maneuvers. Each filter, except the Kalman filter, satisfied the computer requirements. The variances and covariances of the tracking accuracies were calculated automatically by the Kalman and simplified Kalman filters. Monte Carlo analysis verified that these theoretical values were within 15 percent of the observed experimental ("actual") values, so that they could be used directly in kill probability calculations. The Wiener and α-β filters did not automatically provide suitable measures of tracking performance and auxiliary computations would have had to be added if these filters were implemented. It was determined that these extra computations essentially equated the Wiener to the simplified Kalman filter in terms of computer requirements. Based on these results, the simplified Kalman filter was selected for implementation because it provided tracking accuracies within 10 percent of the Kalman filter, because it satisfied the computer requirement constraints, and because it automatically provided accurate tracking accuracy statistics.

VIII. Conclusions

In many practical systems, the constant-gain Wiener filter provides tracking accuracy equivalent to that of the more sophisticated Kalman filter at one third the computational cost. If high accuracy is required and the length of the transient period approaches that of the tracking interval, the simplified Kalman filter becomes attractive for implementation. Furthermore, the Kalman class of tracking filters uniquely provides accurate measures of tracking error statistics, even in the presence of missed data points. Nevertheless, the implementation of extremely simple filters, such as the two-point extrapolator, may be justified, in some cases, by the nature of the system requirements and the impact of tracking performance on these requirements.

The indications of relative filter accuracy and computer requirements, as shown in Figs. 3 and 4 for the five tracking filters considered in this paper, can also be used to bound the performance and requirements of other filters, such as stored-gain Kalman filters [8], piecewise constant-gain linear filters, least-squares filters, and polynomial filters. For example, the first two of these classes of filters, if designed carefully, provide accuracies between that provided by the Wiener filter (which is itself a

	Prediction Accuracy (1σ)				Percent of Computer Time Devoted to Tracking	Auxiliary Functions
	Range (yd)	Bearing (deg)	Speed (Knot)	Course (deg)		
System Requirements	150	0.80	2.5	7.0	10	Kill Probability Calculation
Two Point Extrapolator	210	1.04	3.7	9.7	0.8	–
Wiener Filter	142	0.71	2.4	6.8	2.4	–
α-β Filter	173	0.80	2.8	8.2	5.2	–
Simplified Kalman Filter	138	0.65	2.4	6.5	9.5	*
Kalman Filter	130	0.63	2.1	6.1	12	*

*Tracking accuracy statistics suitable for the auxiliary functions are calculated automatically.

Fig. 6. Filter and system parameters for the second example.

stored-gain filter, as well as the simplest type of piecewise constant-gain filter) and the real-time Kalman filter. Their computer requirements differ from that of the Wiener filter primarily in the number of gain vectors to be stored and in the countdown logic necessary to properly utilize these gains. Consequently, these filters will present a computational burden between those provided by the Wiener and α-β filters.

The performance summary and filter selection rationale that appear in this paper provide a basis from which a first cut tracking filter selection and approximate estimates of filter accuracies and computer requirements can quickly be made for a large number of systems that must track piloted vehicles with wide ranges of maneuver capabilities. When more reliable figures are required for a specific system, a tracking simulation will generally be necessary. A general design procedure for adaptive, rather than fixed structure, tracking filters [9], [10] is presently being investigated, and it is hoped that results in this area will be available subsequently to augment the study presented here.

Appendix

Whitening the Maneuver Noise

The z transform of the correlation function of a discrete signal plays a role analogous to the power spectral density of a continuous signal. If $U(z)$ is the z transform of an input to a filter whose impulse response is $H(z)$, then the output signal $w(k)$ has the transform $W(z) = U(z)H(z)$. Further, as shown in [5], if $\Phi_{uu}(z)$ is the z transform of the correlation function $r(k)$ of the input signal, then the output has a correlation function $c(k)$ which is the inverse transform of $\Phi_{ww}(z) = \Phi_{uu}(z) \cdot H(z)H(1/z)$.

The process of whitening the maneuver noise is begun by associating with it an appropriate correlation function. A satisfactory correlation function, in the sense of providing realistic approximations to the statistics of the maneuver, as well as yielding mathematically tractable solutions, is

$$r(k) = \sigma_M{}^2 \rho^{|k|}. \tag{13}$$

In this equation, $\sigma_M{}^2$ is the maneuver variance and ρ is the correlation coefficient ($|\rho| \leqslant 1$) modelled by

$$\rho = \frac{E[u(k)u(k-1)]}{\sigma_M{}^2} = \begin{cases} 1-\lambda T, & T \leqslant \frac{1}{\lambda} \\ 0, & T > \frac{1}{\lambda} \end{cases}. \tag{14}$$

The quantity λ is essentially the inverse of the average maneuver duration and is suitably chosen to reflect the target's dynamic maneuver characteristics. This correlation model is analogous to that in [3], which has the discrete time form $r^1(k) = \sigma_M{}^2 e^{-\alpha|k|T}$, where α is the inverse of the continuous time maneuver time constant of the target. Hence, ρ equals $e^{-\alpha T}$, and when αT is small, ρ can be approximated closely by $1-\alpha T$, so that α and λ become identical.

This correlation function assures that as time between maneuver samples increases, the correlation between these samples decreases. The two extreme cases occur when ρ is unity and when ρ approaches zero. The former case represents the completely correlated case, namely when $u(k)$ is held constant for all samples. This corresponds physically to a constant acceleration. The latter case is the completely uncorrelated case, namely white noise. Here the maneuver at one sampling period is completely uncorrelated with the maneuver at a different sampling period. This situation prevails when the target exhibits constant velocity except for random disturbances.

The z transform of the maneuver correlation function $r(k)$ is

$$\Phi_{uu}(z) = \sigma_M{}^2 \sum_{k=-\infty}^{\infty} \rho^{|k|} z^k = \frac{\sigma_M{}^2(1-\rho^2)}{(1-\rho z)(1-\rho/z)}. \tag{15}$$

A suitable whitening filter is given by

$$H(z) = \frac{1-\rho z}{z}. \quad (16)$$

With this filter, the z transform of the output correlation function is given by

$$\Phi_{ww}(z) = \sigma_M^2 \frac{1-\rho^2}{(1-\rho z)(1-\rho/z)}$$

$$\cdot \frac{1-\rho z}{z} \cdot \frac{1-\rho/z}{1/z} = \sigma_M^2(1-\rho^2) \quad (17)$$

so that the output signal $w(k)$ is white noise and has a correlation function

$$c(k) = \begin{cases} \sigma_M^2(1-\rho^2) & k=0 \\ 0 & \text{otherwise.} \end{cases} \quad (18)$$

The input-output relation for this filter is

$$H(z) = \frac{1-\rho z}{z} = \frac{W(z)}{U(z)} \quad (19)$$

so that $u(k)$ may be expressed recursively in terms of the white noise sequence $w(k)$ by

$$u(k+1) = \rho u(k) + w(k). \quad (20)$$

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