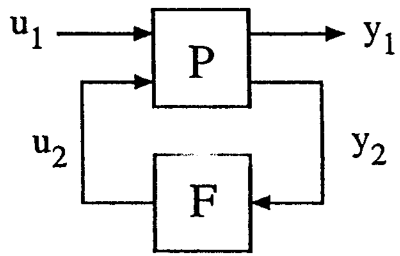
H_∞ Control With Full State Feedback



H_∞ Control With Full State Feedback

- Find a constant state-feedback matrix F to stabilize P and to achieve closed-loop gain

$$\|F_l(P,F)\|_\infty < \gamma \quad \text{where}$$

γ = requirement → Lower to find optimum (iteration procedure)

$$P(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ I & 0 & 0 \end{array}\right]$$

and where

(*i*) (A, B_1) is stabilizable

(*ii*) (C_1, A) is detectable

(*iii*) (A, B_2) is stabilizable

(*iv*) $D_{12}^T C_1 = 0$

(*v*) $D_{12}^T D_{12} = I$

H_∞ Control With Full State Feedback

- For full rank D_{12}, we can scale so that $D_{12}^T D_{12} = I$.
- Significance of the cross-weight requirement $D_{12}^T C_1 = 0$?
  ($D_{12}^T \to u_1$, $C_1 \to x$) dependant {R, Q weights equivalent}
- Result by Doyle, Glover, Khargonekar, and Francis (1988).

H_∞ Control With Full State Feedback

- Compare this to the structure of the LQR problem:

$$P = \left[\begin{array}{c|cc} A & I & B \\ \hline Q^{1/2}F & 0 & 0 \\ 0 & 0 & R^{1/2} \\ I & 0 & 0 \end{array}\right] = \left[\begin{array}{c|cc} A & I & B_2 \\ \hline C_1 & 0 & D_{12} \\ I & 0 & 0 \end{array}\right]$$

- In LQR, the equivalent cross-weight relation is

$$D_{12}^T C_1 = \begin{bmatrix} 0 & R^{1/2} \end{bmatrix} \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix} = 0$$

H_∞ Control With Full State Feedback

- In H_∞, C_1 plays the role of Q in LQR, and D_{12} plays the role of R in LQR.
- Both LQR and this version of H_∞ state feedback require that the external output y_1 have two orthogonal parts:
  - a state component $C_1 x$
  - a control component $D_{12} u_2$
- A modified version of this H_∞ result can handle cross-weightings $D_{12}^T C_1 \neq 0$.

H_∞ Control With Full State Feedback

- The solution is

input u_1 influence ↑

$$X = \mathbf{Ric}\begin{bmatrix} A & \frac{1}{\gamma^2}B_1B_1^T - B_2B_2^T \\ -C_1^TC_1 & -A^T \end{bmatrix}$$

$$F = -B_2^TX$$

- If the Hamiltonian is in **dom Ric** (that is, if it has no imaginary eigenvalues), then

$$F_l(P,F) \quad \text{is stable}$$

$$\|F_l(P,F)\|_\infty < \gamma$$

H_∞ Control With Full State Feedback

- Current H_∞ results do not give a unique optimal solution: instead they give solutions with $\| \cdot \|_\infty < \gamma$.
- One must iterate on γ to approach the optimal solution:

$$\begin{aligned} \text{imaginary } \lambda_i(H) \quad &\Rightarrow \quad \gamma \text{ too small} \\ \text{no imaginary } \lambda_i(H) \quad &\Rightarrow \quad \gamma \text{ too large} \end{aligned}$$

H_∞ Control With Full State Feedback

- For full-rank D_{12}, we can scale so that $D_{12}^T D_{12} = I$
- Factor D_{12} using the SVD:

$$D_{12} = U_1 \begin{bmatrix} 0 \\ \Sigma_1 \end{bmatrix} V_1^T$$

where $U_1^T U_1 = I$, $V_1^T V_1 = I$, and Σ_1 is diagonal.

- Insert $V_1 \Sigma_1^{-1}$ in series with the control signal u_2. The scaled $\hat{D}_{12}$ matrix is

$$\hat{D}_{12} = D_{12} V_1 \Sigma_1^{-1} = U_1 \begin{bmatrix} 0 \\ \Sigma_1 \end{bmatrix} V_1^T V_1 \Sigma_1^{-1} = U_1 \begin{bmatrix} 0 \\ I \end{bmatrix}$$

so that

$$\hat{D}_{12}^T \hat{D}_{12} = [0 \quad I]\, U_1^T U_1 \begin{bmatrix} 0 \\ I \end{bmatrix} = I$$

H_∞ Control With Full State Feedback

- Define $R^{-1} = V_1 \Sigma_1^{-2} V_1^T$. Then

$$\begin{aligned} D_{12}^T D_{12} &= V_1 \begin{bmatrix} 0 & \Sigma_1 \end{bmatrix} U_1^T U_1 \begin{bmatrix} 0 \\ \Sigma_1 \end{bmatrix} V_1^T \\ &= V_1 \Sigma_1^2 V_1^T \;=\; R \end{aligned}$$

- The scaled H_∞ solution is

$$\begin{aligned} X &= \mathbf{Ric} \begin{bmatrix} A & \frac{1}{\gamma^2} B_1 B_1^T - B_2 R^{-1} B_2^T \\ -C_1^T C_1 & -A^T \end{bmatrix} \\ F &= -R^{-1} B_2^T X \end{aligned}$$

- As $\gamma \to \infty$, this becomes identical to the LQR solution!

H_∞ Optimal Control: General Solution

- For years after the introduction of H_∞ in early 1980s, no practical, low-order solution algorithm was available.
- Known methods required an elaborate series of state-space operations:
    - Youla parameterization
    - Inner-outer factorization
    - Spectral factorization
    - Hankel-optimal approximation

H_∞ Optimal Control: General Solution

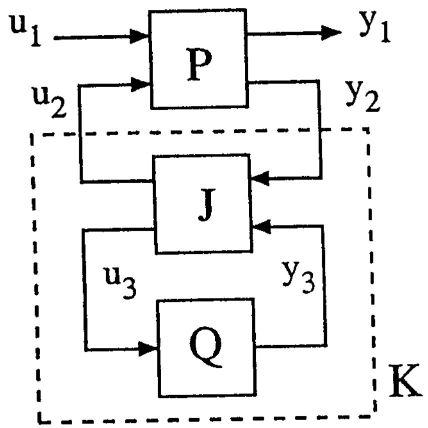
- Each step added more states. The controller could have 10 to 30 times more states than the plant!
- Model reduction (e.g., balancing truncation) helped; but situation was still awkward.
- Breakthrough in 1987-88: H_∞ solutions with two Riccati equations and an n-state controller.
- Work by Doyle, Glover, Khargonekar, Francis, Limebeer, Kasenally, Safonov, Chiang, Tadmor, Bryson, and others.

H_∞ Optimal Control: General Solution

- Parallels are pervasive between the new H_∞ results and H_2 (LQG):

  - n-state controller
  - state feedback – state estimator
  - two Riccati equations
  - Q parameterization of all
    controllers giving norm $< \gamma$
  - H_∞ central controller
    becomes H_2 optimal as $\gamma \to \infty$

H_∞ Optimal Control: General Solution



$$P = \begin{array}{c} \\ \\ y_1 \\ y_2 \end{array} \left[\begin{array}{c|cc} & u_1 & u_2 \\ A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] = \begin{array}{c} \\ y_1 \\ y_2 \end{array} \begin{array}{cc} u_1 & u_2 \\ \left[\begin{array}{c} P_{11} \\ P_{21} \end{array}\right. & \left.\begin{array}{c} P_{12} \\ P_{22} \end{array}\right] \end{array}$$

H_∞ Optimal Control: General Solution

- Make the following assumptions for $P(s)$:

  (i) (A, B_1) and (A, B_2) are stabilizable

  (ii) (C_1, A) and (C_2, A) are detectable

  (iii) $D_{12}^T D_{12} = I$

  (iv) $D_{21} D_{21}^T = I$

  (v) $D_{11} = 0$ and $D_{22} = 0$

- Requirements (iii) and (iv) imply that

$$\begin{aligned}(\text{no. of outputs in } y_1) &\geq (\text{no. of actuators in } u_2)\\ (\text{no. of inputs in } u_1) &\geq (\text{no. of sensors in } y_2)\end{aligned}$$

H_∞ Optimal Control: General Solution

- The solution is:

$$\begin{aligned}
X &= \mathbf{Ric}\begin{bmatrix} A - B_2 D_{12}^T C_1 & \gamma^{-2} B_1 B_1^T - B_2 B_2^T \\ -\tilde{C}_1^T \tilde{C}_1 & -(A - B_2 D_{12}^T C_1)^T \end{bmatrix} \\
Y &= \mathbf{Ric}\begin{bmatrix} (A - B_1 D_{21}^T C_2)^T & \gamma^{-2} C_1^T C_1 - C_2^T C_2 \\ -\tilde{B}_1 \tilde{B}_1^T & -(A - B_1 D_{21}^T C_2) \end{bmatrix} \\
\tilde{B}_1 &= B_1 (I - D_{21}^T D_{21}) \\
\tilde{C}_1 &= (I - D_{12} D_{12}^T) C_1 \\
F &= -(B_2^T X + D_{12}^T C_1) \\
H &= -(Y C_2^T + B_1 D_{21}^T) \\
Z &= (I - \gamma^{-2} Y X)^{-1}
\end{aligned}$$

State Feedback

State Estimator

→ New term

H_∞ Optimal Control: General Solution

- five conditions must hold:

  (*i*) $X \geq 0$ — Positive Semidefinite

  (*ii*) $Y \geq 0$ — "

  (*iii*) $\rho(XY) < \gamma^2$ — largest eigenvalue of XY

  (*iv*) Hamiltonian for X in **dom Ric**

  (*v*) Hamiltonian for Y in **dom Ric**

H_∞ Optimal Control: General Solution

- The parameterization $K = F_l(J, Q)$ of all stabilizing controllers satisfying

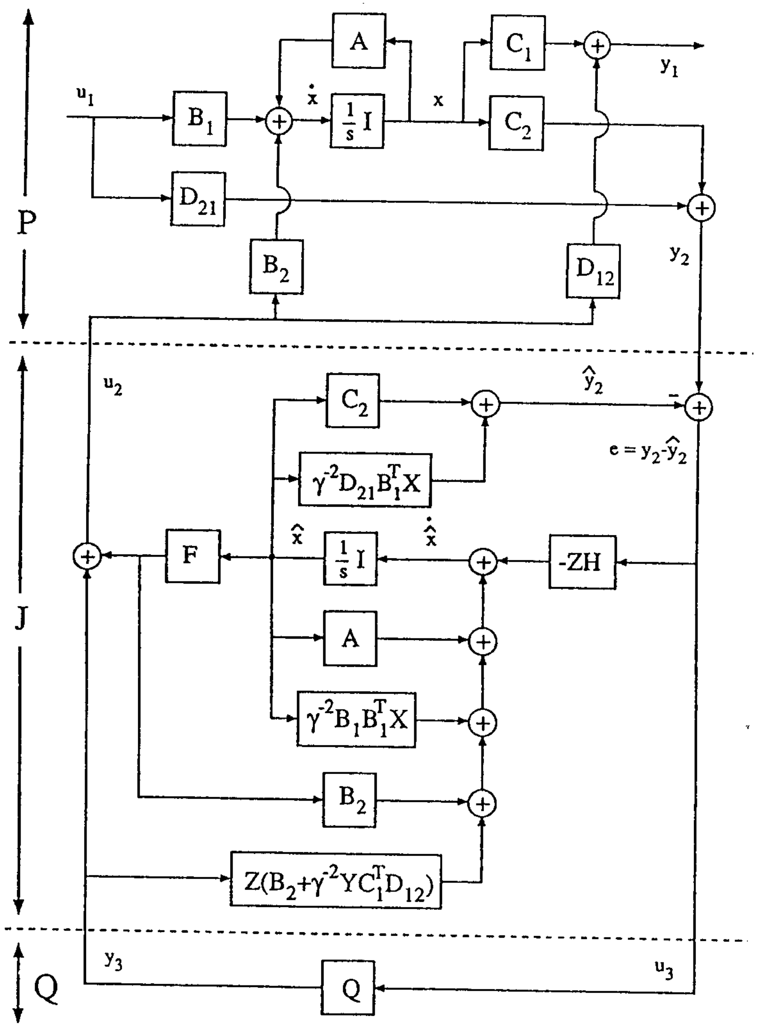
$$\|F_l(P, K)\|_\infty < \gamma$$

is given by all stable Q with $\|Q\|_\infty < \gamma$, where

$$J = \left[\begin{array}{c|cc} \begin{array}{c} A+B_2F+\gamma^{-2}B_1B_1^TX \\ +ZH(C_2+\gamma^{-2}D_{21}B_1^TX) \end{array} & -ZH & Z(B_2+\gamma^{-2}YC_1^TD_{12}) \\ \hline F & 0 & I \\ -(C_2+\gamma^{-2}D_{21}B_1^TX) & I & 0 \end{array}\right]$$

$$= \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

H_∞ Controller Structure



H_∞ Controller Structure

- The controller contains an estimate $\hat{x}$ of the plant's state vector x.
- The output estimate is $\hat{y}_2 = (C_2 + \gamma^{-2} D_{21} B_1^T X)\hat{x}$. The term with γ^{-2} does not appear in LQG.
- The estimated state vector's dynamics (when $Q = 0$) are

estimate of u_1 influence

$$\dot{\hat{x}} = A\hat{x} + \gamma^{-2} B_1 B_1^T X \hat{x} + B_2 u_2 - ZH(y_2 - \hat{y}_2)$$

The term with γ^{-2} and the Z matrix do not appear in LQG.

H_∞ Controller Structure

- The actuator input command is $u_2 = F\hat{x}$: full-state feedback on the estimated states $\hat{x}$.
- The free parameter Q acts as a gain between the output estimation error $y_2 - \hat{y}_2$ and the actuator input u_2.
- As $\gamma \rightarrow \infty$, the controller structure becomes identical to LQG!

Differences Between H_∞ and H_2

- The Hamiltonians for H_∞ are identical to those for H_2, except for the terms $\gamma^{-2} B_1 B_1^T$ and $\gamma^{-2} C_1^T C_1$ in the upper right corners.
- Because B_1 doesn't appear in the H_2 state feedback ARE, H_2 state feedback cannot account for the way disturbances enter the plant. H_∞ can.
- Because C_1 doesn't appear in the H_2 state estimator ARE, H_2 state estimation cannot account for the way the states influence the output. H_∞ can.

Differences Between H_∞ and H_2

- H_2 state estimators cannot trade off the accuracy of individual state estimates against each other. This is a root cause of poor LQG robustness.

- H_∞ chooses from a wider set of control laws than H_2: should be able to deliver higher performance and robustness levels.