RADIO ASTRONOMY
CORE IDEAS: LECTURE 3— POLARISATION OF ELECTROMAGNETIC FIELDS

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1. Polarisation

1.1. Basic ideas.

- Polarization of EM waves of fundamental importance — not least in radio astronomy. Polarization — and changes in polarisation — are important parameters.
- Man-made signals generally have definite polarisation (completely polarised) but radio astronomy signals may be only partially polarised or entirely unpolarised.
- Plane wave polarised in xy plane, travelling in $+\hat{\mathbf{z}}$ directions:

$$E_x = E_1 \sin(\omega t - \beta z) \tag{1}$$

$$E_y = E_2 \sin(\omega t - \beta z + \delta) \tag{2}$$

with E_1 and E_2 constant amplitudes; $\beta = 2\pi/lambda$; δ is phase difference.
- Vector given by:

$$\mathbf{E} = \hat{\mathbf{x}}E_x + \hat{\mathbf{y}}E_y \tag{3}$$

- To investigate polarisation, set $z = 0$; after some manipulation to eliminate t, find expression for *polarisation ellipse*:

$$a{E_x}^2 - bE_xE_y + c{E_y}^2 = 1 \tag{4}$$

with

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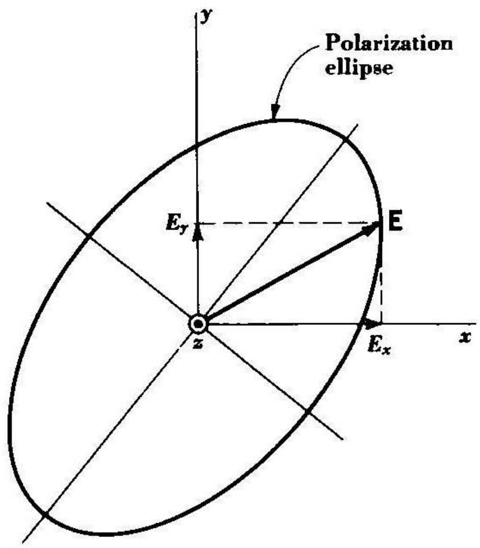

FIGURE 1. Relation of instantaneous electric-field vector **E** to polarisation ellipse, after [1, Fig. 4-1].

$$a = \frac{1}{{E_1}^2 \sin^2 \delta} \tag{5}$$

$$b = \frac{2\cos\delta}{E_1 E_2 \sin^2 \delta} \tag{6}$$

$$c = \frac{1}{{E_2}^2 \sin^2 \delta} \tag{7}$$

- This is eq. for ellipse.
- *Axial ratio* $AR \equiv$ OA/OB; τ is *tilt angle*.
- Some special cases:
  - $E_2 = 0$; linearly polarized in $\hat{\mathbf{x}}$.
  - $E_1 = 0$; linearly polarized in $\hat{\mathbf{y}}$.
  - $E_1 = E_2, \delta = 0°$; slant linearly polarized.
  - $E_1 = E_2, \delta = \pm 90°$; circularly polarised.
    $\delta = +90°$; left (hand) CP
    $\delta = -90°$; right (hand) CP
- Polarisation handedness (IEEE, formerly IRE, definition): freeze space; looking along direction of propagation, direction of motion of tip of **E** defines handedness.

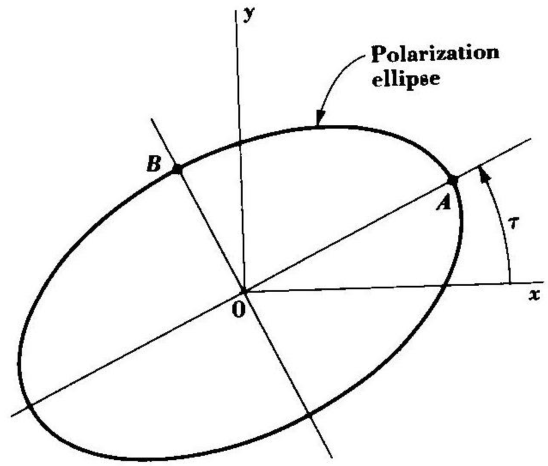

FIGURE 2. Polarisation ellipse geometry, after [1, Fig. 4-2].

- Note that older, physics, notation, was opposite; there, freezing time, the opposite handed helix was traced out in space.
- IEEE convention almost exclusively used nowadays — has advantage that geometrical handedness of actual helical antenna corresponds with this convention.
- Here, general elliptical polarisation was defined i.t.o. two linearly polarised components.
- Can also use two oppositely handed circularly polarised waves of different amplitudes. Useful when studying Faraday rotation.

1.2. **Poincaré sphere.**

- The *Poincaré sphere* is a method for visualising *polarisation state.*
- W.r.t. Eqs. 1 and 2, let:

$$\gamma = \arctan \frac{E_2}{E_1},\ 0 \leq \gamma \leq 90^\circ \tag{8}$$

with γ being the amplitude ratio, τ the tilt angle ($0 \leq \tau \leq 180^\circ$)and ϵ

$$\epsilon = \operatorname{arccot}(\mp \mathrm{AR}),\ -45^\circ \leq \tau \leq 45^\circ \tag{9}$$

with the axial ratio $\mathrm{AR} = \frac{\text{major axis}}{\text{minor axis}}$, $1 \leq |\mathrm{AR}| \leq \infty$; minus for RH and plus for LH CP.

- Polarisation state can be defined by either γ and δ, or ϵ and τ.

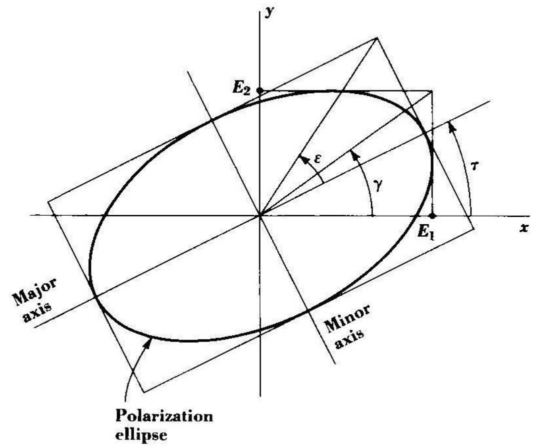

Fig. 4-5. **Relation of amplitudes E_1 and E_2 and angles ϵ, γ, and τ to the polarization ellipse.**

FIGURE 3. Relations of amplitudes E_1 and E_2 and angles ϵ, γ and τ to the polarisation ellipse, after [1, Fig. 4-5].

- Interrelated as:

$$\cos 2\gamma = \cos 2\epsilon \cos 2\tau \tag{10}$$

$$\tan \delta = \frac{\tan 2\epsilon}{\sin 2\tau} \tag{11}$$

or

$$\tan 2\tau = \tan 2\gamma \cos \delta \tag{12}$$

$$\sin 2\epsilon = \sin 2\gamma \sin \delta \tag{13}$$

- Let polarisation state i.t.o. γ and δ be $P(\gamma, \delta)$, and similarly for ϵ and τ, $M(\epsilon, \tau)$.
- On the Poincaré sphere, the angles are given as follows, with reference to Fig. 4:
- $M(\epsilon, \tau)$:
  2ϵ: latitude, $-90° \leq 2\epsilon \leq 90°$;
  2τ: longitude, $0° \leq 2\tau \leq 360°$.
- $P(\gamma, \delta)$:
  2γ: great circle distance from origin, $0° \leq 2\gamma \leq 180°$;
  δ: angle of great circle line w.r.t. equator, $-180° \leq \delta \leq 180°$.
- Special cases of interest:

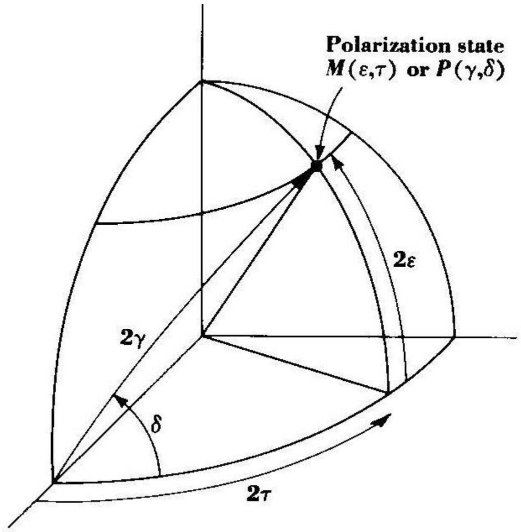

FIGURE 4. Poincaré sphere, after [1, Fig. 4-6].

$\delta = 0°$or$180°$: Lies on equator; E_x and E_y exactly in or out of phase. At origin, linear and horizontal ($\tau = 0°$); at 90° to right, linear, 45°; at 180° linear and vertical.

$\delta = \pm 90°$ and $E_2 = E_1$ ($2\gamma = 90°$): Lies at poles. North pole: LHCP; south pole: RHCP.

- In general, northern hemisphere is LHEP, southern hemisphere is RHEP.

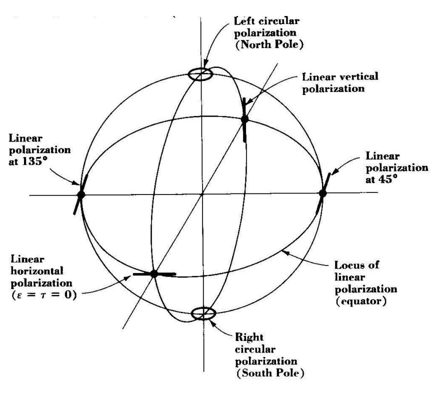

Fig. 4-7. Polarization at cardinal points of Poincaré sphere.

FIGURE 5. Polarisation at cardinal points on Poincaré sphere

1.3. **Response of an antenna to a wave of arbitrary polarisation.**

- For a linearly polarised dipole antenna, ℓ being effective length and polarisation, to an incident field $\mathbf{E}$, antenna terminal voltage V is given by:

$$V = \mathbf{E} \cdot \hat{\ell} \tag{14}$$

- Graphically:

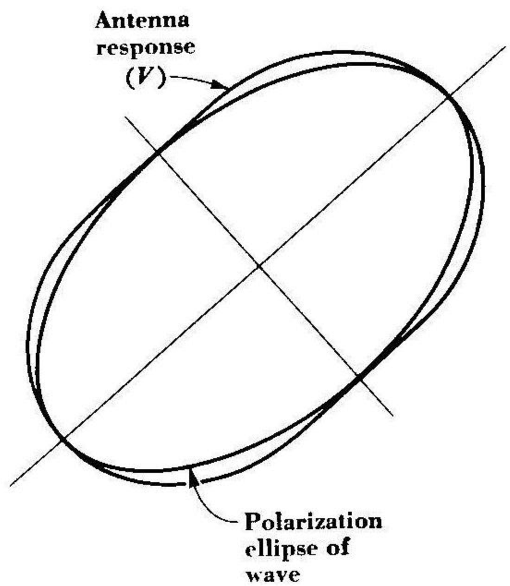

FIGURE 6. Response of a linearly polarised antenna to an elliptically polarised wave, after [1, Fig. 4-9]..

- More generally:

$$V = E\ell \frac{\cos MM_a}{2} \tag{15}$$

with MM_a great circle distance between points on the P. sphere.
- For example: wave LHCP, antenna RHCP, $MM_a = 180°$, thus $V = 0$. In general, antenna is blind to wave of antipodal state.

1.4. **Partial polarisation and the Stokes parameters.**

- Discussion to here assumes completely polarised wave, i.e. E_1, E_2 and δ are constant w.r.t. time.

- Typical celestial radio sources result from superposition of large number of statistically independent wave of varying polarisations:

$$E_x = E_1(t)\sin\omega t \tag{16}$$
$$E_y = E_2(t)\sin[\omega t + \delta(t)] \tag{17}$$

- Such a wave could be produced artificially as shown in 7

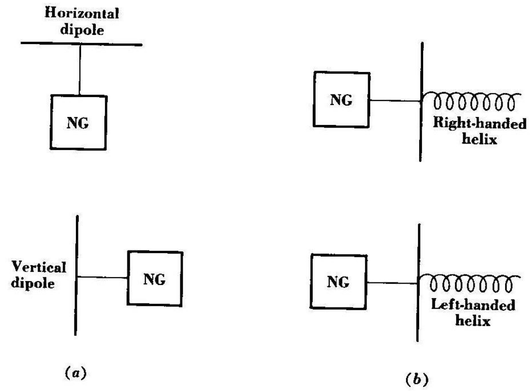

Fig. 4-10. Methods of generating a completely unpolarized wave. At (*a*) two independent noise generators are connected to vertical and horizontal dipoles (wave out of page). At (*b*) two independent noise generators are connected to helical-beam antennas of opposite hand (wave to right).

FIGURE 7. Methods of generating a completely unpolarised wave. At (a) two independent noise generators are connected to vertical and horizontal dipoles. (b) is similar using two helical antennas. After [1, Fig. 4-10].

- Most general situation involves *partial polarisation*, for which Stokes parameters are the standard radio astronomy representation.

1.4.1. *The Stokes parameters (Sir George Stokes, 1852) for completely polarised waves.*

- Starting by considering definition and application to a *completely* polarised wave.
- W.r.t. Fig. 8: by considering fields in the (x', y') and (x, y) to reference axes (see [1, Section 4.4] for details),and the x and y components of Poynting vector[1], $S_x = {E_1}^2/Z$ and $S_y = {E_2}^2/Z$, the four Stokes parameters are defined as:

[1]Assuming that E_1 and E_2 are r.m.s. values.

$$I = S = S_x + S_y = \frac{{E_1}^2}{Z} + \frac{{E_2}^2}{Z} \tag{18}$$

$$Q = S_x - S_y = \frac{{E_1}^2}{Z} - \frac{{E_2}^2}{Z} = S \cos 2\epsilon \cos 2\tau \tag{19}$$

$$U = (S_x - S_y) \tan 2\tau = S \cos 2\epsilon \sin 2\tau = 2\frac{E_1 E_2}{Z} \cos(\delta_1 - \delta_2) \tag{20}$$

$$V = (S_x - S_y) \tan 2\epsilon \sec 2\tau = S \sin 2\epsilon = 2\frac{E_1 E_2}{Z} \sin(\delta_1 - \delta_2) \tag{21}$$

- Follows that:

$$I^2 = Q^2 + U^2 + V^2 \tag{22}$$

and that

$$\frac{U}{Q} = \tan 2\tau \tag{23}$$

and

$$\frac{V}{S} = \sin 2\epsilon \tag{24}$$

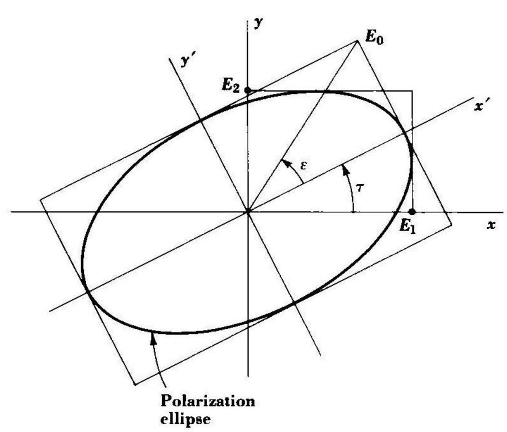

Fig. 4-11. Relation of polarization-ellipse axes (x',y') to reference axes (x,y).

FIGURE 8. Relation of polarisation-ellipse axes (x', y') to reference axes (x, y), after [1, Fig. 4-11].

- Some special cases:
  LHCP: $S_x = S_y$, AR = 1, $\epsilon = 45°$ (τ need not be specified): $I = S$, $Q = U = 0$, $V = S$.
  RHCP: $S_x = S_y$, AR = 1, $\epsilon = -45°$ (ditto τ): $I = S$, $Q = U = 0$, $V = -S$.
  LP with $\tau = 0°$: $S_x = S$, $S_y = 0$, AR $= \infty$, $\epsilon = 0°$: thus $I = S$, $Q = S$, $U = V = 0$.

1.4.2. *Stokes parameters for completely unpolarised or partially polarised waves.*

- For this case, time averages must be taken, indicated by $< \ldots >$:

$$I = \frac{< {E_1}^2 >}{Z} + \frac{< {E_2}^2 >}{Z} = S_x + S_y = S \tag{25}$$

$$Q = \frac{< {E_1}^2 >}{Z} - \frac{< {E_2}^2 >}{Z} = S_x - S_y = S < \cos 2\epsilon \cos 2\tau > \tag{26}$$

$$U = \frac{2}{Z} < E_1 E_2 \cos\delta >= S < \cos 2\epsilon \sin 2\tau > \tag{27}$$

$$V = \frac{2}{Z} < E_1 E_2 \sin\delta >= S < \sin 2\epsilon > \tag{28}$$

and

$$I^2 \geq Q^2 + U^2 + V^2 \tag{29}$$

where $\delta = \delta_1 - \delta_2$.

- Stokes parameters have following interpretation:
  I: total power, or sum of two autocorrelation functions.
  Q: Difference of power in x and y components, or difference of two autocorrelation functions
  U and V: Cross correlation functions.
- For a completely unpolarised wave, $S_x = S_y$, so $I = S$, $Q = U = V = 0$.
- Degree of polarisation d defined as ratio to polarised to total power, i.e.:

$$d = \frac{\sqrt{Q^2 + U^2 + V^2}}{I};\ 0 \leq d \leq 1 \tag{30}$$

- Stokes parameters can also be normalised (see see [1, Section 4.4] for details).
- Similarly, LHCP and RHCP waves can also be used as the basis (ditto).
- If two wave have identical Stokes partakers, the waves are also indentical.
- Also, if two independent waves propagating in same direction are superimposed, the Stokes parameters are also the sum of the individual ones.

1.5. Polarisation measurements.

- Modern realtime DSP methods make computation of the Stokes parameters straightforward, assuming both polarisations are measured (which requires antennas supporting two polarisations).
- Conceptually, this can be measured by simply rotating a dipole; with $W_\|$ and $W_\perp$ the maximum and minimum reposes respectively, the degree of linear polarisation d_ℓ is given by:

$$d_\ell = \frac{|W_\| - W_\perp|}{W_\| + W_\perp};\ 0 \leq d_\ell \leq 1 \tag{31}$$

- Similar expressions hold for circular polarisation.
- A scheme for measuring the four normalised Stokes parameters using only power responses is outlined in [1, Section 4.6]. Since modern interferometric systems are all coherent (phase-preserving), this is mainly of historical interest.

References

[1] J. D. Kraus, *Radio Astronomy*. New York: McGraw-Hill, 1966.