2D Electronmagnetic Analysis of a Three-Phase AC Machine

1 Introduction

The purpose of this assignment is to calculate the 2D magnetic field analysis of a simple three-phase electrical machine and compare that with the simplified 1D magnetic field analysis af the same machine. This is also a practical application of the knowledge obtained in:

- Systems and Signals 244 – w.r.t. the Fourier series expansion
- Engineering Mathematics 242 – w.r.t. the solving of partial differential equations
- Applied Matematics B 242 – w.r.t. vector analyusis
- Electromagnetics 314 – w.r.t. electromagnetic boundary conditions
- Energy Systems 344 – w.r.t. 1D electromagnetic analysis of rotating electrical machines

2 Simple Three-Phase Machine

In Figure 1 below, the structure of a simple two pole three-phase machine is shown with a single, concentrated, overlapping coil configuration per phase. For our comparative analysis, we will use the value for the machine as given in Table 1.

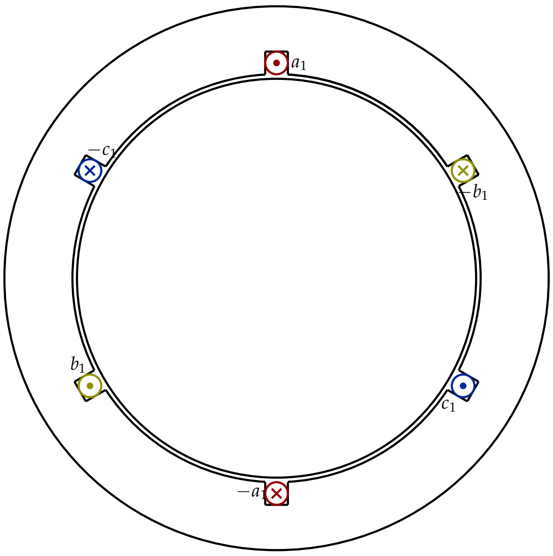

Figure 1: A two pole three-phase machine with a single, concentrated, overlapping coil configuration per phase

For the 1D electromagnetic field analysis, we first calculated the Fourier series expansion of the MMF for one phase, e.g. phase a, as shown in Figure 2. Then we calculated the Fourier series expansion of the combined three-phase rotating MMF. We assumed that the effect of the stator slots were negligible, i.e. that the stator slots were infinitely thin.

3 Surface Current Density Distribution

For the 2D electromagnetic field analysis, a similar approach will be followed. Now however, we are going to make use of the surface current density distribution, K, on the surface of the stator. First we

Name		Value	Description	Unit
p	=	1	pole pairs	
N	=	100	number of turns per coil	[turns]
I_m	=	$\sqrt{2} \cdot 10$	stator current amplitude	[A]
r_n	=	100E-3	nominal radius measured to centre of the air-gap	[m]
h_y	=	50E-3	stator iron thickness	[m]
μ_{rr}	=	1000	rotor iron relative permeability	
μ_{rs}	=	1000	stator iron relative permeability	
g	=	2E-3	air-gap length	[m]

Table 1: The different machine parameters.

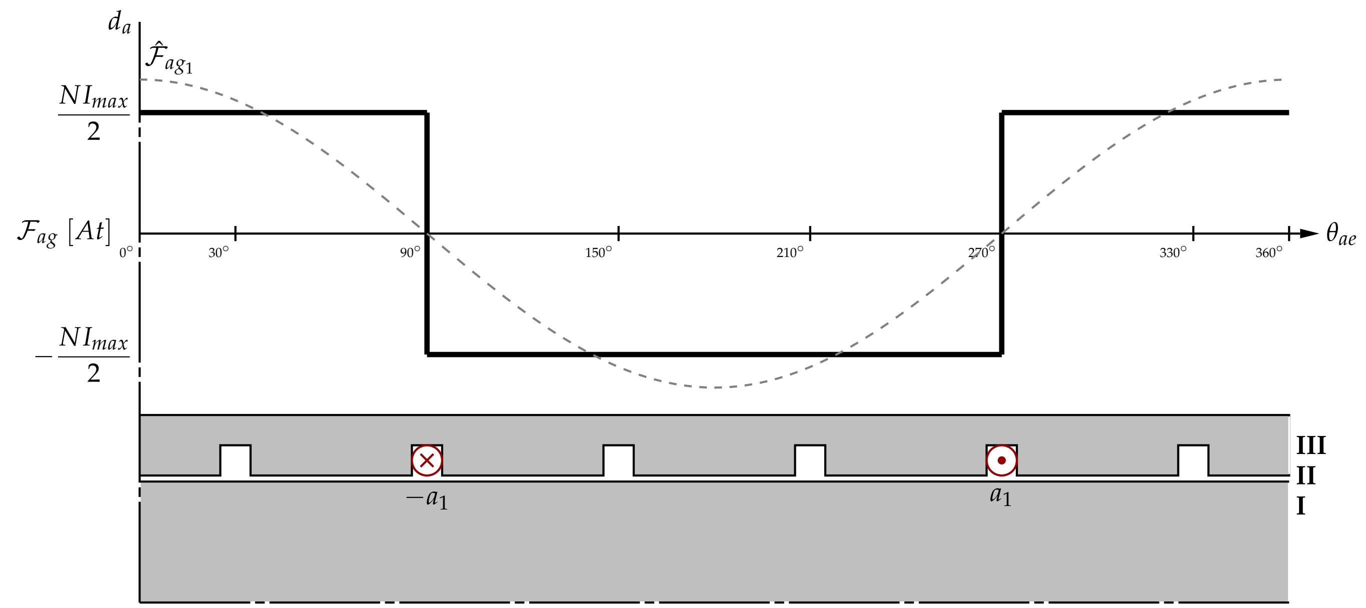

Figure 2: A linear representation of a two pole three-phase machine, showing only the winding layout of phase a together with its associated, ideal MMF distribution.

will calculate the surface current density for each phase separately and then the combined rotating surface current density distribution for all three phases together. Again we will assume that the stator slot width tend to be infinity thin. To calculate the surface current density distribution, we first need to calculate the conductor density distribution.

Although it was not specifically stated whilst performing our 1D analysis, we actually assumed that the conductor density distribution consist of impulses with strength $\frac{N}{r}$ at each stator slot. The conductor density distribution for phase a, $n_a(\theta_{ae})$, measured in [turns/m], is graphically depicted in Figure 3. For our perfectly symmetrical machine, the conductor density distribution for phase b and c, can thus be given by the following two equations:

$$n_b(\theta_{ae}) = n_a(\theta_{ae} - 120^\circ) \tag{1}$$
$$n_c(\theta_{ae}) = n_a(\theta_{ae} + 120^\circ) \tag{2}$$

From the conductor density distribution, the surface current density distribution of phase a, $K_a(\theta_{ae}, t)$, measured in [At/m], can easily be calculated, by simply multiplying the conductor density distribution of

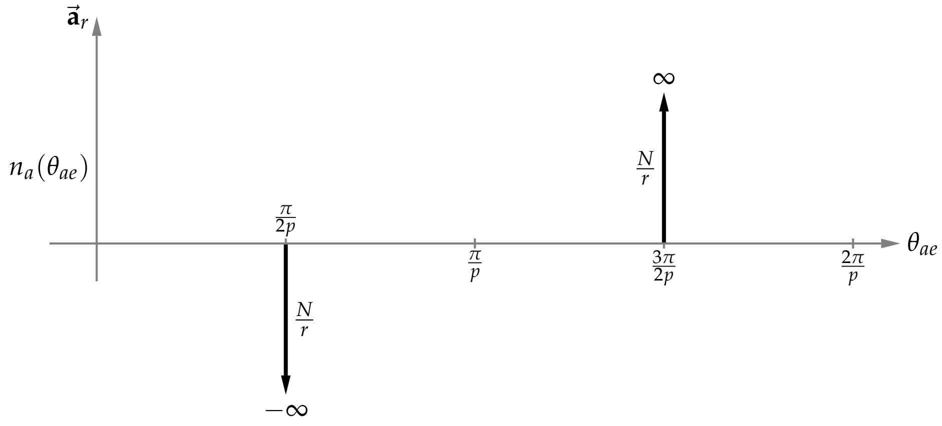

Figure 3: The conductor density distribution for phase *a*.

each phase with the phase current flowing in that phase, so that we can write,

$$K_a(\theta_{ae}, t) = n_a(\theta_{ae}) i_a(t) \tag{3}$$

$$K_b(\theta_{ae}, t) = n_a(\theta_{ae} - 120^\circ) i_b(t) \tag{4}$$

$$K_c(\theta_{ae}, t) = n_a(\theta_{ae} + 120^\circ) i_c(t) \tag{5}$$

and the combined *rotaing* surface current density distribution for all three phases as,

$$K_{abc}(\theta_{ae}, t) = K_a(\theta_{ae}, t) + K_b(\theta_{ae}, t) + K_c(\theta_{ae}, t) \tag{6}$$

To aid in the 2D magnetic field analysis, we divide the machine into different regions, as also shown in Figure 2, with:

Region I – the disk or circular shaped rotor region

Region II – the annulus shaped air-gap region

Region III – the annulus shaped stator region

Furthermore, we define the boundary radii between these regions as:

$$r_i = r_n - \frac{g}{2} \quad \text{– The inner boundary between Region I \& II} \tag{7}$$

$$r_{ii} = r_n + \frac{g}{2} \quad \text{– The inner boundary between Region II \& III} \tag{8}$$

$$r_{iii} = r_n + \frac{g}{2} + h_y \quad \text{– The outer boundary of Region III} \tag{9}$$

Thus with the surface current density distribution on the surface of the stator region, i.e. on the boundary between region II and III, we can use the radius value of r_{ii} for our surface current density distribution. The surface current densitity distribution of phase *a*, $K_a(\theta_{ae}, t)$, is graphically depicted in Figure 4. Comparing Figure 4 with Figure 2 we can see that the MMF distribution is actually the integral of the surface current density distribution, so that in general we can write:

$$\mathcal{F}_{ag} = \int K_a(\theta_{ae}, t) r_{ii} d\theta_{ae} \tag{10}$$

The calculation of the combined *rotating* surface current density distribution, $K_{abc}(\theta_{ae}, t)$, is very similar to that done for the combined rotating MMF distribution used in the 1D electromagnetic field analysis and will have the following form:

$$K_{abc}(\theta_{ae}, t) = \sum_{n=1,5,7,\ldots}^{\infty} \hat{K}_{abc_n} \cdot \begin{cases} \sin(\theta_{ae} - \omega_e t) & \text{for the positive rotating space vector components} \\ \sin(\theta_{ae} + \omega_e t) & \text{for the negative rotating space vector components} \end{cases} \tag{11}$$

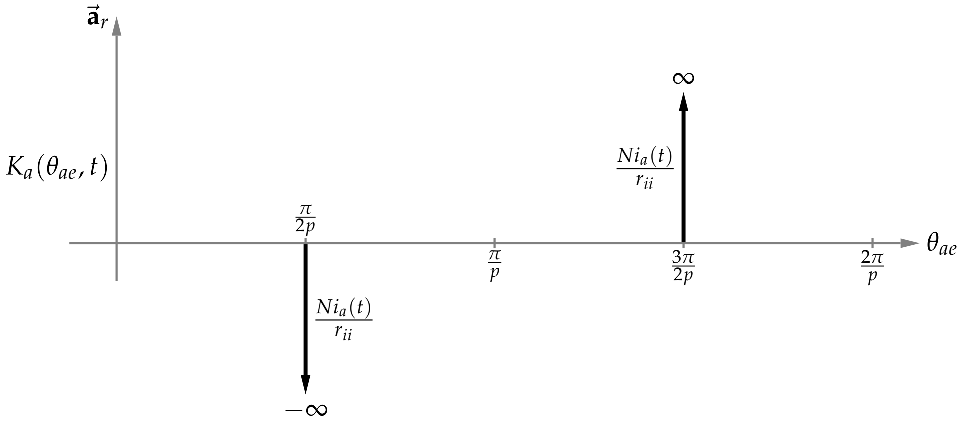

Figure 4: The surface current density distribution for phase a.

4 Sub-Domain Analysis Method using the Vector Potential

With our electrical machine already divided into regions, or sub-domains, we will make use of the magnetic vector potential, $\mathbf{A}$, to obtain the 2D magnetic flux-density, $\mathbf{B}$, in the machine. The governing equation to solve the magnetic vector potential in each of the machines regions or sub-domains, is given by the following Poisson equation:

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J} \tag{12}$$

However, because we assume that the stator slots are infinity thin and that there is only a surface current flowing in between regions II & III, each of the three regions can be considered to be a "current free" region. This implies that (12) reduces to the simple Laplace form of:

$$\nabla^2 \mathbf{A} = 0 \tag{13}$$

The different regions of the three-phase machine, together with their boundaries, relative permeability and governing equation, is summarised in Table 2.

Region	Range for r	μ_r	Governing equation
III	$r_{iii} \geq r \geq r_{ii}$	μ_{rs}	$\nabla^2 \mathbf{A} = 0$
II	$r_{ii} \geq r \geq r_i$	1	$\nabla^2 \mathbf{A} = 0$
I	$r_i \geq r \geq 0$	μ_{rr}	$\nabla^2 \mathbf{A} = 0$

Table 2: The different regions of the three-phase machine, together with their boundaries, relative permeability and governing equation.

5 Interior Dirichlet Boundary Problem Solution

This is a normal interior Dirichlet boundary problem. The solution to the magnetic vector potential in the different regions will only consist of a z-component, $A_z(r, \theta)$. Also, the solution to the magnetic vector potential for the Laplace equation, will only consist of a general solution, i.e. no particular solution. This solution can be obtained by making use of the separation of variables technique. It can be proven that the general solution to the interior Dirichlet problem in the Disk-shaped region I, will have the following form:

$$A_z(r, \theta) = \sum_{n=1}^{\infty} r^{np} \left(a_n \cos(np\theta) + b_n \sin(np\theta) \right) \tag{14}$$

with

$$\begin{aligned} n &= \text{the space harmonic number and} && (15)\\ p &= \text{the number of pole pairs} && (16) \end{aligned}$$

The general solution to the interior Dirichlet problem in the Annulus-shaped regions II & III, will have the following form:

$$A_z(r,\theta) = \sum_{n=1}^{\infty} \left(a_n r^{np} + b_n r^{-np}\right)\cos(np\theta) + \left(c_n r^{np} + d_m r^{-np}\right)\sin(np\theta) \quad (17)$$

These solutions can be simplified further, by noting that the although there is no forcing function, the solution should have the same "shape" as the surface current density, $K_{abc}(\theta_{ae}, t)$. The general solution to the surface current density was already given in (11). Thus, the simplified general solution to the interior Dirichlet problem in the Disk-shaped region I, will be:

$$A_z(r,\theta) = \sum_{n=1,5,7,\ldots}^{\infty} b_n r^{np} \cdot \begin{cases} \sin(np\theta - \omega_e t) & \text{for the positive rotating space vector components} \\ \sin(np\theta + \omega_e t) & \text{for the negative rotating space vector components} \end{cases} \quad (18)$$

With the simplified general solution to the interior Dirichlet problem in the Annulus-shaped regions II & III, given by:

$$A_z(r,\theta) = \sum_{n=1,5,7,\ldots}^{\infty} \left(c_n r^{np} + d_n r^{-np}\right) \cdot \begin{cases} \sin(np\theta - \omega_e t) & \text{for the positive rotating space vector components} \\ \sin(np\theta + \omega_e t) & \text{for the negative rotating space vector components} \end{cases} \quad (19)$$

6 Finding the Solution to the Magnetic Flux Density

With the solution of the magnetic vector potential known, the magnetic flux density can be easily calculated, by noting that:

$$\begin{aligned} \vec{\mathbf{B}} &= \nabla \times \vec{\mathbf{A}} && (20)\\ &= \frac{1}{r}\begin{vmatrix} \vec{\mathbf{a}}_r & r\vec{\mathbf{a}}_\theta & \vec{\mathbf{a}}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix} && (21)\\ &= \frac{1}{r}\left(\frac{\partial A_z}{\partial \theta} - r\frac{\partial A_\theta}{\partial z}\right)\vec{\mathbf{a}}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\vec{\mathbf{a}}_\theta + \frac{1}{r}\left(\frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right)\vec{\mathbf{a}}_z\,. && (22) \end{aligned}$$

With the magnetic vector potential only having a z-component, the magnetic flux density reduces to,

$$\begin{aligned} \vec{\mathbf{B}}(r,\theta) &= \frac{1}{r}\frac{\partial A_z}{\partial \theta}\vec{\mathbf{a}}_r - \frac{\partial A_z}{\partial r}\vec{\mathbf{a}}_\theta && (23)\\ &= B_r\vec{\mathbf{a}}_r + B_\theta\vec{\mathbf{a}}_\theta\,. && (24) \end{aligned}$$

From the above it can be seen that the radial, or r-component, $B_r(r,\theta)$ and the azimuthal or θ-component, $B_\theta(r,\theta)$, of the magnetic flux density, can be calculated directly from the magnetic vector potential as shown below.

$$\begin{aligned} \therefore \quad B_r(r,\theta) &= \frac{1}{r}\frac{\partial A_z}{\partial \theta} && (25)\\ B_\theta(r,\theta) &= -\frac{\partial A_z}{\partial r} && (26) \end{aligned}$$

From the magnetic flux density, the radial, or r-component, $H_r(r,\theta)$ and the azimuthal or θ-component, $H_\theta(r,\theta)$, of the magnetic field intensity, can easily be calculated, as shown below.

$$\therefore \quad H_r(r,\theta) = \frac{B_r}{\mu_0 \mu_r} \tag{27}$$

$$H_\theta(r,\theta) = \frac{B_\theta}{\mu_0 \mu_r} \tag{28}$$

7 Boundary Conditons

With the solution of $A_z(r,\theta)$, $B_r(r,\theta)$ and $H_\theta(r,\theta)$ known in all three regions in terms of the coefficients, b_n^I, c_n^{II}, d_n^{II}, c_n^{III} and d_n^{III}, we therefore require only five boundary value equation to solve these five unknown coefficients.

7.1 Using the Radial Component of the Magnetic Flux Density

On the boundary, $r = r_i$, between region I and region II as well as on the boundary, $r = r_{ii}$, between region II and region III, we can use the fact that in the cylindrical coordinate system, the radial magnetic flux density components on the boundary between two region will be the same.

$$B^I_{r|r=r_i} = B^{II}_{r|r=r_i} \tag{29}$$

$$B^{II}_{r|r=r_{ii}} = B^{III}_{r|r=r_{ii}} \tag{30}$$

This is analogues to the normal component of the magnetic flux density on a boundary between two regions in the rectangular coordinate system, as shown in Figure 5.

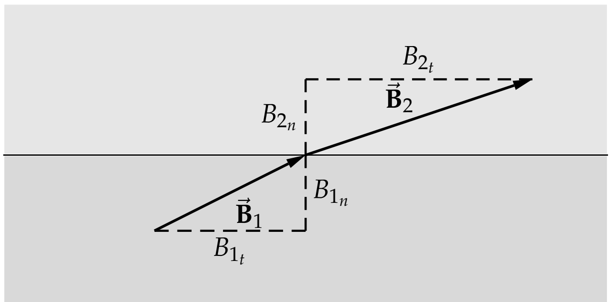

Figure 5: The boundary condition for the normal component of the magnetic flux density.

7.2 Using the Azimuthal Component of the Magnetic Field Intensity

On the boundary, $r = r_i$, between region I and region II we can also use the fact that in the cylindrical coordinate system, the azimuthal (or tangential) magnetic field intensity components on the boundary between two region will be the same. However, on the boundary, $r = r_{ii}$, between region II and region III, the difference in the azimuthal (or tangential) component of the magnetic field intensity will be equal to the surface current density on the boundary.

$$H^I_{\theta|r=r_i} = H^{II}_{\theta|r=r_i} \tag{31}$$

$$H^{II}_{\theta|r=r_{ii}} = H^{III}_{\theta|r=r_{ii}} + K_{abc}(\theta_{ae}, t) \tag{32}$$

This is analogues to the tangential component of the magnetic field intensity on a boundary of two regions in the rectangular coordinate system, as shown in Figure 6.

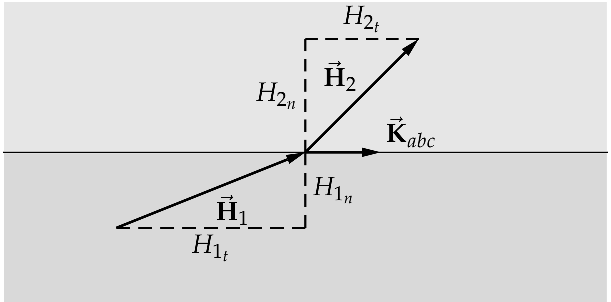

Figure 6: The boundary condition for the tangential component of the magnetic field intensity.

7.3 Using the Magnetic Vector Potential

If we were to assume that no magnetic flux "escapes" through the outer boundary of region III, we can say that the magnetic vector potential on the outer boundary of region III is equal to zero, i.e.:

$$A^{III}_{z|r=r_{iii}} = 0 \tag{33}$$

8 Assignment Report to Complete

Write a short report in LaTeX[1] or MS Word[2] clearly showning the mathematical derivation of points 1 to 3a as well as the calculation of point 7 below. Then write a short script in Python or Matlab to perform the calculations and plotting necessary for points 3b to 7 below. Include your script as an appendix in your report. Both the report in PDF format, as well as your script as a single `*.py` or `*.m` file should be emailed to me. Use your student number for both filenames and ES344-TASK-I in the subject line. The handin date and time is **14 September 2015 @ 09:00**

1. Calculated the surface current density distribution for each phase, i.e. $K_a(\theta_{ae}, t)$, $K_b(\theta_{ae}, t)$ and $K_b(\theta_{ae}, t)$ and then the three-phase surface current density distribution, $K_{abc}(\theta_{ae}, t)$.

2. From the simplified general solutions of the magnetic vector potential, A_z as given in (18) for region I and from (19) for regions II and III, make use of (25) and (28) to calculate the general solution of the radial magnetic flux density, B_r and the tangential magnetic field intensity component, B_θ for each region.

3. Make use of the boundary conditions as presented in (29), (30), (31), (32) and (33), to solve the coefficients, b^I_n, c^{II}_n, d^{II}_n, c^{III}_n and d^{III}_n up to the 49$^{\text{th}}$ harmonic.

   (a) First construct the following matrix equation in symbolic form,

   $$\left[C_n\right]\left[X_n\right] = \left[D_n\right] \tag{34}$$

   with (say)

   $$\left[X_n\right] = \begin{bmatrix} b^I_n \\ c^{II}_n \\ d^{II}_n \\ c^{III}_n \\ d^{III}_n \end{bmatrix} \tag{35}$$

[1] Nice typsetting of the math, but with a steep learning curve.
[2] Not so nice typsetting of math but with almost no learning curve.

(b) Then substitute in the value from Table 1 make use of Python or Matlab's matrix capabilities to solve the coefficients up to the 49$^{\text{th}}$ harmonic. Please note that we do not need to solve for even or triplet harmonics.

4. From the solution to the coefficients above to, make use of Python or Matlab's 2D contour plotting capabilities to:

   (a) plot the magnetic vector potential, A_z, for the whole machine in 2D @ $t = 0$.

   (b) also with the radial magnetic flux density component, B_r, as well as the azimuthal magnetic field intensity, H_θ known, plot the magnitude of the magnetic flux density in 2D, @ $t = 0$, with:

$$B_\theta = \mu_0 \mu_r H_\theta \tag{36}$$

   and

$$B_{mag} = \sqrt{B_r^2 + B_\theta^2} \tag{37}$$

5. Plot the radial component of the magnetic flux density distribution in the centre of the air-gap, $B_r(r, \theta)$ for $\theta = 0^\circ \ \ldots \ 360^\circ$ and with $r = r_n$ @ $t = 0$.

6. Repeat the calculation in step 3b, but now with $\mu_{rr} = \mu_{rs} = 100\,000$ and repeat step 5 above.

7. Finally, with μ_{rr} & $\mu_{rs} \rightarrow \infty$ calculate and plot the magnetic flux density distribution in the centre of the air-gap using the 1D analysis method up to the 49$^{\text{th}}$ harmonic @ $t = 0$.

   (a) How does the waveforms of the magnetic flux density distribution obtained using the 1D and 2D analysis methods compare with one another.

   (b) How does the fundamental component of the magnetic flux density distribution obtained using the 1D and 2D analysis methods compare with one another.