Solve a discrete difference equation with complex roots

DDE: $y(k) - 2\,y(k-1) + 4\,y(k-2) = 4\,r(k) - 3\,r(k-1) + 2\,r(k-2)$

Determine $y(k)$ closed form equation with $r(k) = \mu(k)$ a unit step

① Z-transform of DDE:

$$\left(1 - 2z^{-1} + 4z^{-2}\right) Y(z) = \left(4 - 3z^{-1} + 2z^{-2}\right) R(z)$$

$$\Rightarrow Y(z) = \frac{4z^2 - 3z + 2}{z^2 - 2z + 4} R(z) = \frac{z}{z-1} \cdot \frac{4z^2 - 3z + 2}{z^2 - 2z + 4} \rightarrow \text{Complex roots}$$

② Do partial fraction expansion of $\frac{Y(z)}{z}$:

$$\Rightarrow \frac{Y(z)}{z} = \frac{r_1}{z-1} + \frac{r_2 z + r_3}{z^2 - 2z + 4} = \frac{4z^2 - 3z + 2}{(z-1)(z^2 - 2z + 4)} \quad - (A)$$

$$r_1 = (z-1)\frac{Y(z)}{z}\bigg|_{z=1} = \frac{4-3+2}{1-2+4} = \frac{3}{3} = 1 \rightarrow$$

$$\therefore \frac{Y(z)}{z} = \frac{z^2 - 2z + 4 + (r_2 z + r_3)(z-1)}{(z-1)(z^2 - 2z + 4)} = \frac{(1+r_2)z^2 + (-2-r_2+r_3)z + (4 - r_3)}{(z-1)(z^2 - 2z + 4)} \quad - (B)$$

③ Compare coefficients (A) and (B):

$z^2: \; 1 + r_2 = 4 \Rightarrow r_2 = 3$

$z^0: \; 4 - r_3 = 2 \Rightarrow r_3 = 2$

$$\Rightarrow Y(z) = \underbrace{\frac{z}{z-1}}_{\text{unit step}} + \frac{3z^2 + 2z}{z^2 - 2z + 4} \rightarrow \text{Sum of exponentially decaying sine and cosine functions}$$

④ We have:

$$a^k \sin \Omega k \longleftrightarrow \frac{(a \sin \Omega) z}{z^2 - (2a\cos\Omega) z + a^2}$$

$$a^k \cos \Omega k \longleftrightarrow \frac{z^2 - (a\cos\Omega) z}{z^2 - (2a\cos\Omega) z + a^2}$$

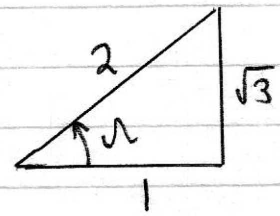
Compare with $Y(z)$ term

$\therefore$

$a^2 = 4 \Rightarrow a = 2$

$\Rightarrow 4\cos\Omega = 2$

$\Rightarrow \cos\Omega = 1/2$



$\Rightarrow \Omega = \frac{\pi}{3}$

$\Rightarrow a \sin\Omega = \sqrt{3}$

$\Rightarrow a\cos\Omega = 1$

Thus,

$$\frac{(a\sin\Omega)z}{z^2-(2a\cos\Omega)z+a^2} = \frac{\sqrt{3}z}{z^2-2z+4}$$

$$\frac{z^2-(a\cos\Omega)z}{z^2-(2a\cos\Omega)z+a^2} = \frac{z^2-z}{z^2-2z+4}$$

$$\Rightarrow \frac{3z^2+2z}{z^2-2z+4} = 3\,\frac{z^2-z}{z^2-2z+4} + \frac{5}{\sqrt{3}}\,\frac{\sqrt{3}z}{z^2-2z+4}$$

(5) Get inverse z-transform of $Y(z)$:

$$y(k) = \underline{\mu(k) + 3.2^k\cos\frac{\pi}{3}k + \frac{5}{\sqrt{3}}.2^k\sin\frac{\pi}{3}k}$$

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