Electromagnetics 833: An introduction to (Radio Frequency) Computational Electromagnetics

The Method of Moments: The Mixed Potential Electric Field Integral Equation FEKO — surface and volumetric MoM treatments

Lecture overview

- The program FEKO
- Electric and magnetic field integral equations
- Fredholm integral equation theory
- Triangular patch vector basis functions.
- The Mixed Potential Electric Field Integral Equation.
- The Galerkin MoM approximation thereof.
- Implementing a Galerkin MoM MPIE EFIE.
- Handling dielectrics
- Equivalence principles — surface and volume.

The program FEKO

- Originates with doctoral work of Ulrich Jakobus in Prof. Landstorfer's *Institut für Hochfrequenztechnik, Univ. Stuttgart*.

- Acronym of German name: "**FE**ldberechnung bei **K**örpern beliebiger **O**berlfäche" (*Field computations involving bodies of arbitrary shape*).

- Incorporates powerful MoM treatment involving piecewise linear triangular functions (Rao, Wilton, Glisson, 1982) for metallic structures.

- Also supports MoM treatment of dielectric structures, using either *surface* or *volumetric* treatment.

Electric and magnetic field integral equations.

- From potential theory (cf. Pocklington eqn.) both EFIE and MFIE can be derived.

- EFIE:

$$\begin{aligned}\vec{n} \times E_{inc}(\vec{r}) &= \vec{n} \times \int_S \Big[jk\eta \vec{J}_S(\vec{r}')G(\vec{r},\vec{r}') \\ &\quad + \frac{\eta}{jk}\{\nabla_s' \cdot \vec{J}_S(\vec{r}')\}\nabla' G(\vec{r},\vec{r}')\Big] \, dS' \; \forall \vec{r}, \vec{r} \in S\end{aligned}$$

  $\nabla' \Rightarrow$ differentiation in *source* coordinates. $\vec{n}$ unit vector on surface S.

- Valid for both closed and open surfaces. In latter case, J_S is average of surface currents on both sides.

MFIE

- Given by:

$$\frac{1}{2}\vec{J}_S(\vec{r}) = \vec{n} \times H^{inc}(\vec{r}) + \vec{n} \times \oint_S \vec{J}_S(\vec{r}') \times \nabla' G(\vec{r}, \vec{r}') dS' \;\; \forall \vec{r}, \vec{r} \in S$$

- Valid *only* for closed surfaces.

- In both EFIE and MFIE, singularities raise delicate issues — require careful treatment.

EFIE, MFIE and integral equation theory

- Mathematically, the EFIE is a *Fredholm integral equation of the first kind* (unknown only in kernel) and the MFIE of the *second kind* (unknown inside *and* outside kernel).

- Fredholm II I.E.'s are generally more stable — this motivated much of the MFIE work.

- Requirement for *closed* surface S is frequently a problem in applied CEM work.

Basis functions for surfaces

- Very widely used basis function is the triangular patch, introduced by Rao, Wilton and Glisson in 1982.

- This basis function is very closely related to the edge-based elements widely used in contemporary finite element analysis.

- Basis function is *vector* in nature (individual scalar components can only be recovered with some manipulation).

- Essential idea is to enforce current continuity over an *edge* of a patch.

Triangular vector basis functions

- Interpolation function used:

$$\vec{f}_n(\vec{r}) = \begin{cases} \frac{l_n}{2A_n^+}\,\vec{\rho}_n^{\,+} & \forall \vec{r} \text{ in } T_n^+ \\ \frac{l_n}{2A_n^-}\,\vec{\rho}_n^{\,-} & \forall \vec{r} \text{ in } T_n^- \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

  $\vec{\rho}_n^{\,\pm}$ is a *local* Cartesian coordinate, referred to relevant free vertex.

- RWG convention: subscripts associate with edges, superscripts with faces.

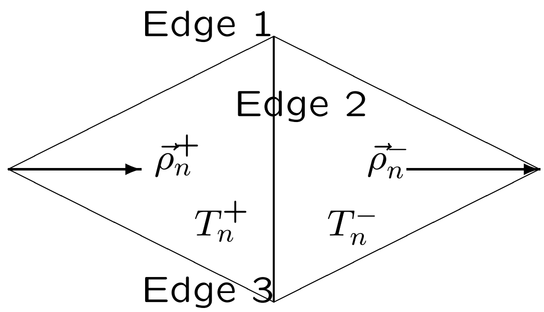
- Current can be expressed as $\vec{J}_n(\vec{r}) = I_n \vec{f}_n(\vec{r})$.

- Can be shown that $\vec{f}_n(\vec{r})$ has no normal component to other edges of $\triangle$ — thus current is normally continuous.

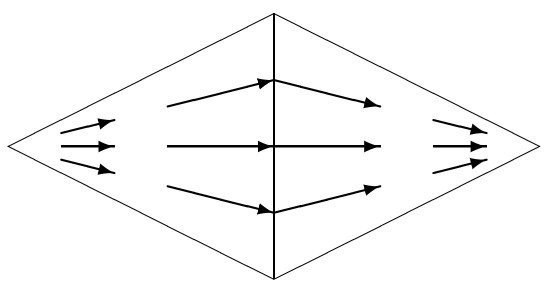
- (From continuity eqn:) charge is thus approximated as piecewise constant.

RWG vector basis functions

- Sketch of two connected triangles T_n^+ and T_n^-, sharing common edge, & supporting an RWG basis function:



- A vector plot of the RWG basis functions:



Derivation of the mixed potential EFIE

- $\vec{E}^{\text{scat}}$ i.t.o. *induced* currents:

$$\vec{E}^{\text{scat}} = -j\omega\vec{A} - \nabla\Phi \tag{2}$$

- Magnetic vector potential:

$$\vec{A}(\vec{r}) = \frac{\mu}{4\pi}\int_S \vec{J}\frac{e^{-jkR}}{R}dS' \tag{3}$$

- Scalar potential:

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon}\int_S \sigma\frac{e^{-jkR}}{R}dS' \tag{4}$$

- Surface charge density σ related to surface divergence of $\vec{J}$ through current continuity:

$$\nabla_s \cdot \vec{J} = -j\omega\sigma \tag{5}$$

Derivation of MPIE EFIE contd.

- Enforce BC on total field, $\hat{n} \times (\vec{E}^{\text{inc}} + \vec{E}^{\text{scat}}) = 0$ on S, to obtain integro-differential equation in $\vec{J}$:

$$-\vec{E}^{\text{inc}}_{\text{tan}} = \left(-j\omega\vec{A} - \nabla\Phi\right)_{\text{tan}}, \ \vec{r} \text{ on } S. \tag{6}$$

- Note derivatives on current in Eq. (5) and on Φ in Eq. (6).

- Requires care with selection of basis and testing functions in MoM development.

- Note that σ associated with RWG basis function is

$$\nabla_s \cdot \vec{f}_n = \begin{cases} \dfrac{l_n}{A_n^+}, & \vec{r} \text{ in } T_n^+ \\ -\dfrac{l_n}{A_n^-}, & \vec{r} \text{ in } T_n^- \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

a.k.a. *pulse doublet*.

MoM formulation for MPIE EFIE

- A Galerkin formulation is adopted — i.e. basis and testing functions the same.
- Surface current approximated as:

$$\vec{J} \approx \sum_{n=1}^{N} I_n \vec{f}_n(\vec{r} \qquad (8)$$

- N is number of *interior* edges.
- On open edges, normal component of $\vec{J}$ is zero; no basis functions needed.
- As usual, n & m are basis functions (source points) & testing functions (field points) respectively.
- Relevant symmetric product defined as usual as

$$< \vec{f}, \vec{g} >= \int_S \vec{f} \cdot \vec{g} dS. \qquad (9)$$

- Eq. (6) is thus tested with $\vec{f}_m$, $m = 1, 2, \ldots ..., N$, yielding

$$< \vec{E}^{\text{inc}}, \vec{f}_m >= j\omega < \vec{A}, \vec{f}_m > + < \nabla\Phi, \vec{f}_m > . \qquad (10)$$

MoM MPIE EFIE: surface charge

- Surface charge in Φ is pulse doublet of Eq. (7).

- Applying ∇ to σ not advisable — results in Dirac delta functions.

- Weak form adopted — move differential operator from source term to testing function.

- Using surface vector calculus identity, last term in Eq. (10) can be rewritten as

$$< \nabla\Phi, \vec{f}_m >= -\int_S \Phi \nabla_S \cdot \vec{f}_m dS. \qquad (11)$$

- Thus differential is moved to the testing function, which is mixed first-order, giving finite result.

MoM MPIE EFIE: approximating outer (testing) integral

- With Eq. (7), integral in Eq. (11) may be written and approximated as follows:

$$\int_S \Phi \nabla_S \cdot \vec{f}_m dS = \ell_m \left(\frac{1}{A_m^+} \int_{T_m^+} \Phi dS \ - \ \frac{1}{A_m^-} \int_{T_m^-} \Phi dS \right)$$
$$= \ell_m \left[\Phi(\vec{r}_m^{c+}) - \Phi(\vec{r}_m^{c-}) \right]. \qquad (12)$$

- Here, average of Φ over each triangle is approximated by the value of Φ at the triangle centroids $c+$ and $c-$.

- Similar approximations are applied to vector potential and incident field terms in Eq. (10), yielding

$$\left\langle \left\{ \begin{array}{c} \vec{E}^{\text{inc}} \\ \vec{A} \end{array} \right\}, \vec{f}_m \right\rangle = \ell_m \left[\frac{1}{2A_m^+} \int_{T_m^+} \left\{ \begin{array}{c} \vec{E}^{\text{inc}} \\ \vec{A} \end{array} \right\} \cdot \vec{\rho}_m^{+} dS \right.$$
$$\left. + \ \frac{1}{2A_m^-} \int_{T_m^-} \left\{ \begin{array}{c} \vec{E}^{\text{inc}} \\ \vec{A} \end{array} \right\} \cdot \vec{\rho}_m^{-} dS \right]$$
$$\approx \frac{\ell_m}{2} \left[\left\{ \begin{array}{c} \vec{E}^{\text{inc}}(r_m^{c^+}) \\ \vec{A}(r_m^{c^+}) \end{array} \right\} \cdot \vec{\rho}_m^{c+} \right.$$
$$\left. + \left\{ \begin{array}{c} \vec{E}^{\text{inc}}(r_m^{c^-}) \\ \vec{A}(r_m^{c^-}) \end{array} \right\} \cdot \vec{\rho}_m^{c-} \right] \qquad (13)$$

- Again, integral over each triangle is eliminated by approximating $\vec{E}^{\text{inc}}$ (or $\vec{A}$) by its value at the triangle centroid.

- With Eqs. (11)–(13), Eq.(10) now becomes:

$$j\omega\ell_m\left[\vec{A}(r_m^{c^+})\cdot\frac{\vec{\rho}_m^{c+}}{2}+\vec{A}(r_m^{c^-})\cdot\frac{\vec{\rho}_m^{c-}}{2}\right]+\ell_m\left[\Phi(r_m^{c^-})-\Phi(r_m^{c^+})\right]$$

$$=\ell_m\left[\vec{E}^{\text{inc}}(r_m^{c^+})\cdot\frac{\vec{\rho}_m^{c+}}{2}+\vec{E}^{\text{inc}}(r_m^{c^-})\cdot\frac{\vec{\rho}_m^{c-}}{2}\right] \quad (14)$$

MoM MPIE EFIE: current approximation

- Substituting the current expansion of Eq. (8) into Eq. (14) yields the usual system of $N \times N$ linear equations of an MoM formulation:

$$\{V\} = [Z]\{I\} \tag{15}$$

- $[Z]$ is an $N \times N$ matrix, and $\{V\}$ and $\{I\}$ are column vectors of length N. Elements of $[Z]$ and $\{V\}$ are given by

$$Z_{mn} = \ell_m \left[j\omega \left(\vec{A}^{+}_{mn} \cdot \frac{\vec{\rho}^{c+}_m}{2} + \vec{A}^{-}_{mn} \cdot \frac{\vec{\rho}^{c-}_m}{2} \right) + \Phi^{-}_{mn} - \Phi^{+}_{mn} \right] \tag{16}$$

$$V_m = \ell_m \left(\vec{E}^{+}_m \cdot \frac{\vec{\rho}^{c+}_m}{2} + \vec{E}^{-}_m \cdot \frac{\vec{\rho}^{c-}_m}{2} \right) \tag{17}$$

where

$$\vec{A}^{\pm}_{mn} = \frac{\mu}{4\pi} \int_S \vec{f}_n(\vec{r}') \frac{e^{-jkR^{\pm}_m}}{R^{\pm}_m} dS', \tag{18}$$

$$\Phi^{\pm}_{mn} = -\frac{1}{4\pi j\omega\epsilon} \int_S \nabla'_s \cdot \vec{f}_n(\vec{r}') \frac{e^{-jkR^{\pm}_m}}{R^{\pm}_m} dS', \tag{19}$$

$$R^{\pm}_m = |\vec{r}^{c\pm}_m - \vec{r}'| \tag{20}$$

and

$$\vec{E}^{\pm}_m = \vec{E}^{\text{inc}}(\vec{r}^{c\pm}_m). \tag{21}$$

MoM MPIE EFIE: integrating over inner (basis) integral

- Potential integrals $\vec{A}(r_m^{c^\pm})$ and $\Phi(r_m^{c^\pm})$ over source triangles must still be computed numerically, using *quadrature*.

- Implementation details in [1, Chapter 6]. Singularity can be side-stepped using appropriate quadrature rule.

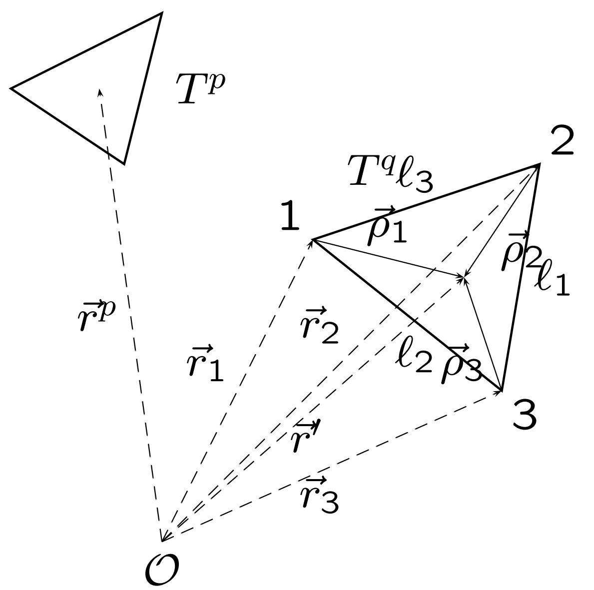
- Galerkin process is computing mutual coupling between two edge-based basis functions, *each* defined on two triangles.

- Thus *four* triangles are involved in the computation of any one matrix entry.

- E.g. $\vec{A}_{mn}^{\pm}$ in Eq. (18) comprises two terms, each computed using quadarature — result of integrating potentials, weighted by basis function, over "source" ΔT_q^+ and ΔT_q^- associated with edge n, for observation point located at the centroid of "testing" ΔT_p^+ or ΔT_p^- associated with edge m.

- These terms are summed in Eq. (16), representing two point (centroid) approximation of $\int$ over the testing ΔT_p^+ and ΔT_p^-.

MoM MPIE EFIE: integrating over inner (basis) integral (contd).



Local coordinates, nodes and edges for source triangle T^q and observation point in testing triangle T^p. Note that all vectors $\vec{r}$ are defined w.r.t. global origin $\mathcal{O}$, but all vectors $\vec{\rho}$ are defined w.r.t. the relevant local free vertex.

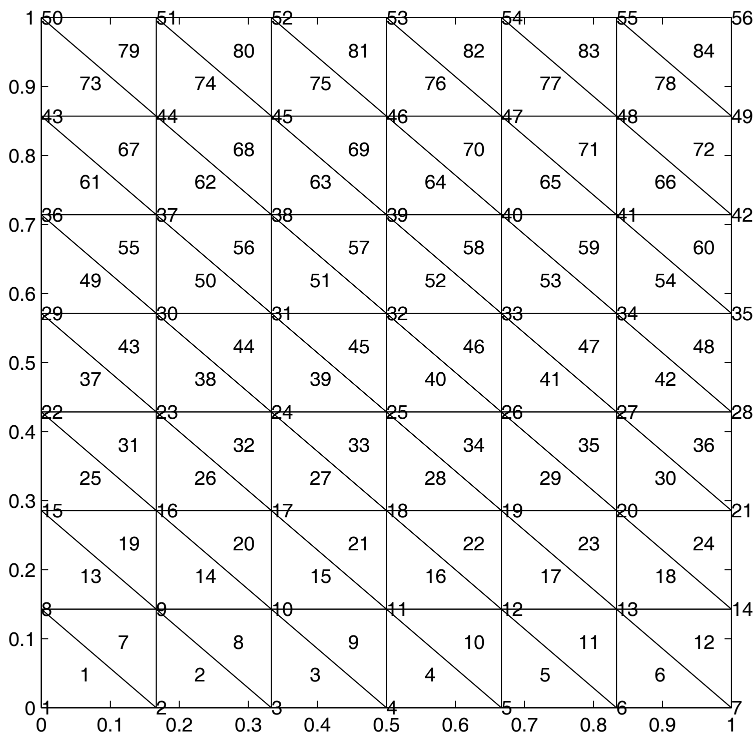
MoM MPIE EFIE in FEKO

- This theory is at the heart of FEKO's handling of PEC surfaces.

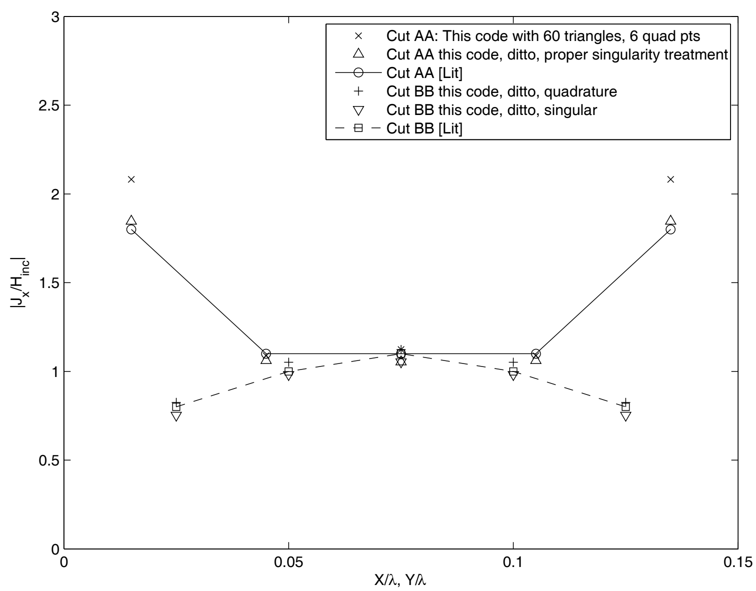
- The integration is done with more sophistication — outer integral also evaluated numerically where needed, proper handling of singular terms.

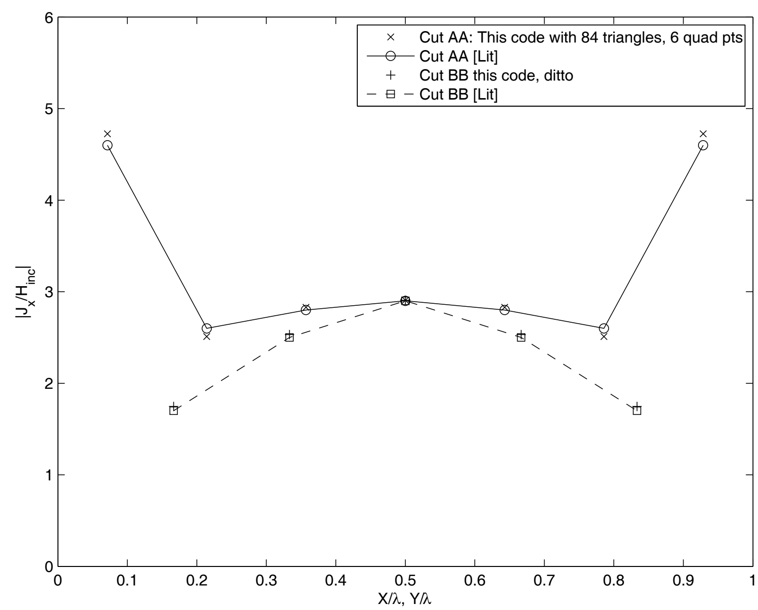
MoM MPIE EFIE: results

An example of a suitable triangular mesh, with $6 \times 7 \times 2 elements$ and 113 degrees of freedom. AA vertical centered cut (constant x); BB horizontal centered cut (constant y).



Top: Dominant component of current on 0.15 λ square flat PEC plate, UPW incident normally x-polarized. Bottom: same, for 1λ plate.





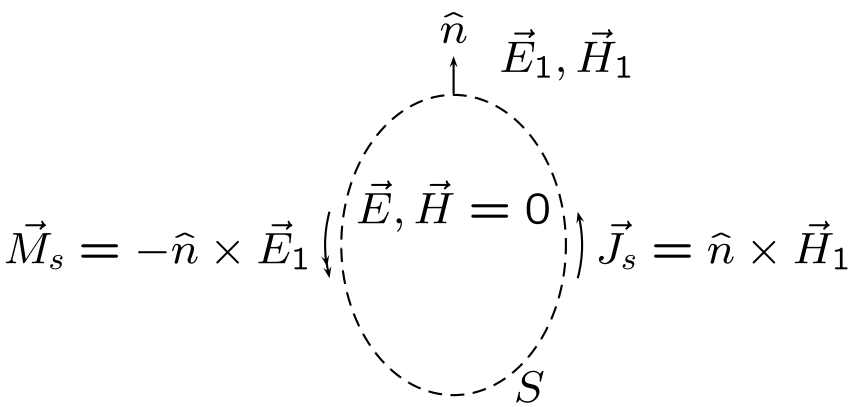
Handling dielectrics

- Rests on equivalence principles.

- Equivalence theorem states that fields in a region of space are uniquely determined by tangential fields over bounding surface.

> A field in a lossy region is uniquely specified by the sources within the region plus the tangential components of the electric field over the boundary, or the tangential components of the magnetic field over the boundary, or the former over part of the boundary and the latter over the rest of the boundary.

[Balanis p. 329]


$\widehat{n}$
$\vec{E}_1, \vec{H}_1$
$\vec{E}, \vec{H} = 0$
$\vec{M}_s = -\widehat{n} \times \vec{E}_1$
$\vec{J}_s = \widehat{n} \times \vec{H}_1$
S

Various forms of surface equivalence

- Since the inner (outer) region does not exist further, it may be replaced with convenient material — often free space. Homogeneous materials sometimes useful, or PEC.

- Typical thin-wire treatment yields actual electric currents — but not essential.

- Homogeneous dielectric regions can be handled thus (and are in FEKO, with both EFIE and MFIE treatments possible).

- Dielectric regions require both equivalent electric and magnetic currents — doubles N, quadruples RAM, run time between 4 and 8 compared to PEC.

- Inhomogeneous dielectric regions require a slightly different approach, using *volume* equivalence.

Volume equivalence

- Equivalent volume electric and magnetic currents are defined to permit the removal of the scatterer and replace it by free space (and thus free space Green's functions):

$$\begin{aligned} \vec{J}_{eq} &= j\omega(\epsilon - \epsilon_0)\vec{E} \\ \vec{M}_{eq} &= j\omega(\mu - \mu_0)\vec{H} \end{aligned} \tag{22}$$

- These *volumetric* currents are discretized using a 3D mesh and solved using the MoM by FEKO.

- This is a *very* computationally expensive formulation $\sim O(f^9)$ and only suitable for very small dielectric regions. FE or FDTD formulation much more appropriate for inhomogeneous dielectrics.

Dielectrics — conclusion

- Homogeneous material structures can be efficiently handled using equivalent surface current formulations. (Both electric and magnetic equivalent currents are required).

- Inhomogeneous material structures can be handled using volumetric MoM, but this is very expensive. FEKO now supports FEM/MoM formulation which is likely to be more efficient.

- Stratified media (eg dielectric half-space) can be very efficiently handled using Sommerfeld potentials. FEKO offers many options in this regard; infinite or finite thickness, grounded or not, sub- or super-strate.

Conclusions

- Lecture has presented background for FEKO's RWG-based rigorous surface modelling (cf. wire-grid modelling) using integral equations.

- Derivation of MPIE EFIE done.

- Outline of approximate Galerkin scheme discussed.

- Equivalence principle has been introduced and application to MoM outlined.

- FEKO implements metallic wire and surfaces using EFIE, dielectric surface equivalence (using either EFIE or MFIE) and dielectric volume equivalence.

- FEKO has a number of other facilities not discussed here, including hybrid asymptotic (PO, UTD) treatments and special Green's functions for a homogeneous or layered dielectric spheres.

References

This lecture is based on :

1 D.B. Davidson, *Computational Electromagnetics for RF and Microwave Engineering*, Cambridge University Press, 2nd edition, Sections 6.1-6.7.