More on one-dimensional FEM: higher-order elements; general boundary conditions

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Computational Electromagnetics

Outline

9.3.1 Higher-order elements - the basics

- Major advantage of FEM over both FDTD and MoM is that it offers straightforward extension to higher-order elements.
- Canonical reference on this for nodal elements is work of Silvester (1970s).
- Will now study a standard 2nd-order element — based on standard Lagrange interpolation polynomials.
- There are other forms — e.g. those based on Hermite interpolation polynomials, which match not just function values but also derivatives.

Higher-order elements — polynomial basis functions

- Usual method of developing 2nd-order element is to insert mid-point node into each element.
- Can be shown that a suitable set of second-order interpolatory polynomials (and hence basis functions) is

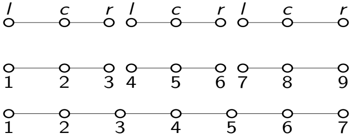
$$\alpha_l = 2\xi_1(\xi_1 - 1/2) \tag{1}$$
$$\alpha_c = 4\xi_1\xi_2 \tag{2}$$
$$\alpha_r = 2\xi_2(\xi_2 - 1/2) \tag{3}$$

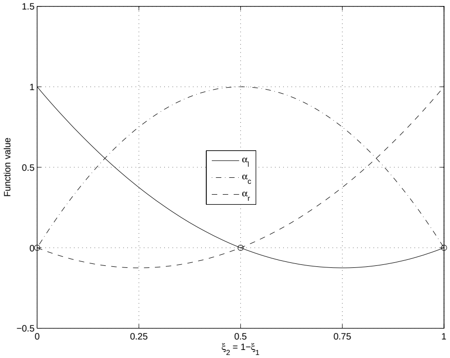
  with the FEM approximation within element e now given by

$$\tilde{V}^e = \alpha_l V_l + \alpha_c V_c + \alpha_r V_r \tag{4}$$

  Subscripts l and r refer as before to left and right nodes; c, to mid-point node.
- Note: α_l is unity at $\xi_1 = 1$ (LH node), & zero at both $\xi_1 = 1/2$ (mid-point node) & $\xi_1 = 0$ (RH node).
- Similarly, α_c interpolates to mid-point node; α_r, to RH node.

Polynomial basis functions *contd.*





Pre-computing the mass and stiffness matrices

- Again: $[S]$ and $[T]$ can be pre-computed. E.g. evaluation of T_{rr}:

$$T_{rr} = \int_0^1 (\alpha_r)^2 \, d\xi = \int_0^1 \xi^2(1-\xi)^2 \, d\xi = \frac{4}{30} \quad (5)$$

- Note that $T_{rr} = T_{ll} \neq T_{cc}$. Using symmetry: four distinct terms to evaluate.
- For more than 2nd-order, symbolic manipulation packages often used.
- Has been done in closed form to reasonably high order in the literature; results for line elements up and including fourth order may be found in [Silvester & Ferrari].
- For second order, results are:

$$[S] = \frac{1}{3}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \quad (6)$$

$$[T] = \frac{1}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} \quad (7)$$

Assembling higher-order mass and stiffness matrices

- Connection matrix approach can be repeated.
- Simpler approach (used in FEM codes) relies on inspection.
- Note that $[T]$, for instance, can be written as:

$$C[\mathcal{C}]^t[\mathcal{T}][\mathcal{C}] = \frac{hC}{6}\begin{bmatrix} 2 & 1 & & & & \\ 1 & 4 & & & & \\ & 1 & 4 & 1 & & \\ & & & \ddots & & \\ & & & 1 & 4 & 1 \\ & & & & 1 & 2 \end{bmatrix}$$

$$= \frac{hC}{6}\begin{bmatrix} T_{ll}^{(1)} & T_{lr}^{(1)} & & & & \\ T_{rl}^{(1)} & T_{rr}^{(1)} + T_{ll}^{(2)} & T_{lr}^{(2)} & & & \\ & T_{rl}^{(2)} & T_{rr}^{(2)} + T_{ll}^{(3)} & T_{lr}^{(3)} & & \\ & & & \ddots & & \\ & & & T_{rl}^{(N-1)} & T_{rr}^{(N-1)} + T_{ll}^{(N)} & T_{lr}^{(N)} \\ & & & & T_{rl}^{(N)} & T_{rr}^{(N)} \end{bmatrix} \quad (8)$$

Assembly *contd.*

- Superscript (n) indicates global element number.
- Wherever nodes are shared between elements, there are contributions to the overall matrix from each element.
- In 1D, the shared nodes are the left and right nodes — true for all orders.
- Diagonal elements add contributions from adjoining elements.
- This extends to higher-order elements.
- This also extends to 2D and 3D (but numbering becomes less regular).

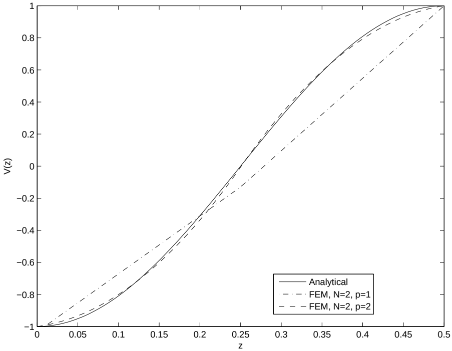
BCs for higher-order elements

- Dirichlet BC enters via forcing vector (RHS of matrix eqn).
- Vector can be shown to be $[\mathcal{M}_{fp}]\{V_p\}$.
- Here, f refers to *free* nodes; p to *prescribed* ones — here, last node on right.
- Sub-matrix $[\mathcal{M}_{fp}]$ is rectangular matrix of dimension $(N-1) \times 1$ (for 1 free node), where N is the number of nodes overall, & vector $\{V_p\}$ is of length 1 here.
- Hence the forcing vector for 2nd order elements, with this particular numbering scheme, becomes

$$\left\{ \begin{array}{c} 0 \\ 0 \\ \vdots \\ -M_{N-2,N}V_N \\ -M_{N-1,N}V_N \end{array} \right\} \tag{9}$$

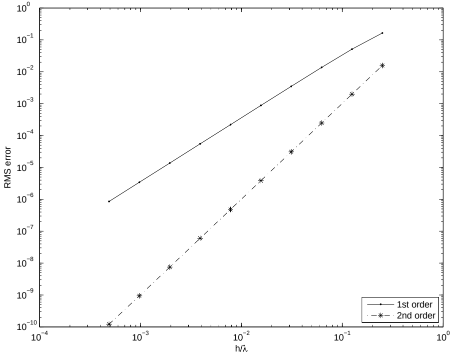
Results

- Results computed with code shown opposite for both 1st and 2nd order.
- Again, $L = C = 1$, so $Z_0 = 1\Omega$, $v_p = 1\text{m/s}$. $f = 1\text{Hz}$, thus $\lambda = 1\text{m}$.
- Now, even with 2 elements, good results already obtained.



Rate of convergence

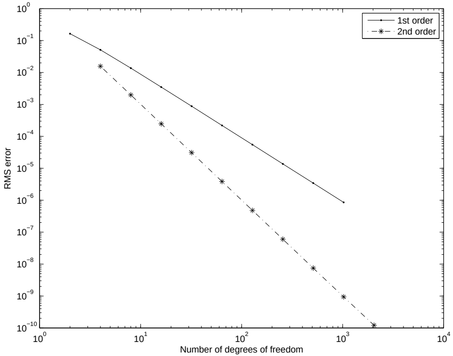
- From theoretical considerations, the expected rate of convergence for p-th order is $\mathcal{O}(h^{p+1})$.



- Slope of curves 1.97 and 3 respectively.

Rate of convergence *contd.*

- Higher-order elements have more degrees of freedom, but even taking this into account, results are still much better:
- In 1D, slope is simply inverse of previous graph's, as $N \propto 1/h$.



More general boundary conditions

- To make FEM generally useful, formulation must be extended to more general BCs.
- Will do this using following DE, the *inhomogeneous scalar Helmholtz equation*:

$$\nabla \cdot (p \nabla u) + k^2 q u = g \tag{10}$$

- u is the scalar variable representing the desired solution.
- Problem is defined on domain Ω; material properties represented by $p(x, y, z)$ and $q(x, y, z)$.
- k^2 is a constant, which may or may not be known.
- $g(x, y, z)$ is a specified driving function.
- BCs specified on surface $\partial\Omega$ which encloses domain.
- For the 1D tx line problem, $u = V$; $p = 1/L$; $q = C$, ; $k^2 = \omega^2$, $g = 0$, and an inhomogenous Dirichlet BC was imposed at $z = \ell$.

More general boundary conditions *contd.*

- BCs are also specified in general form.
- Boundary $\partial\Omega$ consists of three distinct, non-overlapping (but connected) subsets ∂D, ∂N and ∂C which together make up $\partial\Omega$.
- Solution u must satisfy the following conditions on each segment of the boundary:

$$u = u_D \qquad \text{on } \partial D \qquad \text{(Dirichlet)} \tag{11}$$

$$\frac{\partial u}{\partial n} = v_N \qquad \text{on } \partial N \qquad \text{(Neumann)} \tag{12}$$

$$\frac{\partial u}{\partial n} + au = v_C \qquad \text{on } \partial C \qquad \text{(Cauchy)} \tag{13}$$

- Note that normal $\hat{n}$ is by convention outward directed.
- Note also that DEs with higher derivatives need more BCs on each segment.

More general boundary conditions *contd.*

- Equivalent variational functional, whose stationary point corresponds to the solution of Eq. (10), subject to the boundary conditions specified in Eq. (11)–(13), is as follows:

$$\mathcal{F}(u) = \frac{1}{2}\int_{\Omega}\left(p\nabla u \cdot \nabla u - k^2 q u^2 + 2gu\right) + \frac{1}{2}\int_{\partial C} apu^2 dS - \int_{\partial C} puv_C dS - \int_{\partial N} puv_N dS \quad (14)$$

- Note that with a homogeneous Neumann BC (i.e. $v_N = 0$), relevant integral vanishes (as earlier).
- Note also that Dirichlet boundary conditions are imposed explicitly on the solution, & do not appear in variational functional.

An example — a general load

- Using Ohm's law at load end, $V(z=0) = -Z_L I(z=0)$ (note negative sign to be consistent with current flow in the $+z$ direction as in original problem), combined with the appropriate telegraphist's equation, gives

$$\left(\frac{\partial V}{\partial z} - \frac{j\omega L}{Z_L} V \right) \bigg|_{z=0} = 0 \tag{15}$$

- This is a homogeneous Cauchy boundary condition, with $a = +\frac{j\omega L}{Z_L}$ (note that the normal is in the $-z$ direction at the left-hand side of the line, hence $\frac{\partial u}{\partial n} = -\frac{\partial V}{\partial z}$) and $v_C = 0$.
- In practical terms, a term of $ap = \frac{j\omega}{Z_L}$ is added to the first entry in the system matrix. Since $v_C = 0$, RHS vector is not affected.
- With an OC termination, this term vanishes, as expected.
- Similar analysis applies for non-zero internal source impedance (see text for details).

Conclusions

- Lecture has generalized FEA to use 2nd order elements.
- Higher-order FEM routinely used. One limitation (not in 1D) is the elements are rectilinear; analysis for curvilinear elements is available, but pre-computation of S and T matrices is no longer useful.
- Have also demonstrated general BCs for variational functional and shown specific application to general load.
- Reference: DB Davidson, *Computational Electromagnetics for RF and Microwave Engineering*, Cambridge University Press, 2nd edition, 2011, Chapter 9.