A one-dimensional introduction to the FEM

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Computational Electromagnetics

Outline

9.1 Introduction

- Finite element method (FEM) is one of the best-known methods for solution of PDEs.
- Solves a PDE subject to certain boundary values (BVs).
- Originated in contemporary form in structural mechanics during late 1950s.
- 1st specific usage of term “element” due to Courant, 1942.
- First application in electromagnetics in late 1960s.
- FEM (and FDTD) is based on *local* description of *field* quantities — *no* Green’s function as in MoM.
- FEM can use wide variety of elements to approximate geometry (not too relevant in 1D, but important in 2D and 3D).

9.1 Introdution *contd.* - comparison with other CEM methods

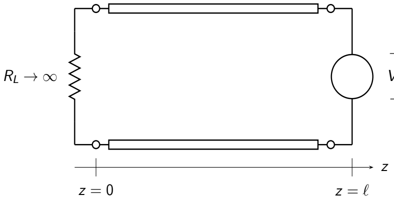
- Main advantage cf. MoM: straightforward handling of materially complex regions.
- Main advantage cf. FDTD: better approximation of geometry ("simplicial" elements — triangles and tetrahedrons) and straightforward extension to higher-order elements.
- Main disadvantage, shared with FDTD cf. MoM: does not include Sommerfeld radiation condition, so some form of mesh termination scheme with a *radiation boundary condition* is needed.
- However, FEM-MoM hybrid addresses this very efficiently.
- Main disadvantage cf. FDTD: more complex implementation, and more difficult to code effectively on parallel computers (eġ. multi-core CPUs, GPUs).
- Recent FDTD-FEM(TD) hybrids address this.

9.2 The variational boundary value formulation

FEM can be derived via two different, but equivalent, procedures:

- Traditional variational approach — the variational boundary value problem (VBVP) — which will be followed here.
- A.k.a. the Ritz, or Rayleigh-Ritz method.
- Other approach: Galerkin weighted residual method formulation — same ideas as in development for MoM.
- Galerkin WRM is more direct at the formulation level, and increasingly used in the CEM literature.
- VBVP is easier for first introduction — allows one to work an element-by-element basis, combining elements into overall system via assembly-by-elements.
- Start with VBVP approach; revisit later using Galerkin.

9.2.1 The model problem



- 1D lossless transmission line.
- Load (on LHS) open ciruited.
- Source (zero internal resistance) on RHS.

- Frequency-domain telegraphist's eqns for V and I along line are:

$$\frac{\partial I(z)}{\partial z} = -j\omega C(z)V(z) \quad (1)$$

$$\frac{\partial V(z)}{\partial z} = -j\omega L(z)I(z) \quad (2)$$

- Note that capacitance and impedance per unit length are permitted to vary along line.
- Eliminating current, and writing as total derivatives (only z remains):

$$\frac{d}{dz}\left(\frac{1}{L}\frac{dV}{dz}\right) + \omega^2 CV = 0$$

9.2.2 The equivalent variational functional

- Full problem to solve is DE (repeated):

$$\frac{d}{dz}\left(\frac{1}{L}\frac{dV}{dz}\right)+\omega^2 CV=0 \tag{3}$$

subject to boundary conditions (BCs):

$$\begin{aligned} V(z=\ell) &= V_0 \\ \left.\frac{dV}{dz}\right|_{z=0} &= 0 \end{aligned} \tag{4}$$

- First BC is inhomogeneous Dirichlet; second is homogeneous Neumann one.
- Latter is derived from Eq. (2), noting that $I(z=0)=0$ for open circuit.
- Without proof at present: stationary point of following functional corresponds to solution of Eq. (3), subject to BCs the boundary conditions of Eq. (4)

$$\frac{1}{2}\int_{\ell}\left[\frac{1}{L}(\frac{dV}{dz})^2-\omega^2 CV^2\right]dz \tag{5}$$

More on the equivalent variational functional

- Note the following: using Eq. (2), functional can be re-written as

$$-\frac{1}{2}\omega^2 \int_\ell \left[LI^2 + CV^2 \right] dz \quad (6)$$

- Two parts of the integrand are respectively stored magnetic and stored electrical energy per unit length.
- Therefore, this is an *energy-related* functional. Distribution of V (and I) such that this quantity is rendered *stationary* — minimized, maximized or a point of inflexion.
- Note also that V of Eq. (3) must be *twice*-differentiable, whereas V in Eq. (5) need only be differentiable *once*.
- This relaxation leads to name "weak form" for VBVP approach.

9.2.2 The finite element approximation of the functional — some basics

- FEM approximates $V(z)$ in *functional* rather than in DE.
- Firstly, approximate the geometry: here, simple line segments — elements.
- Then, approximation the function with an ensemble (set) of functions defined on the elements.
- These elemental functions must be at least once differentiable.
- To satisfy continuity requirements, approximation should be continuous at element boundaries.
- Finally, set of elements must also satisfy overall boundary conditions.

FEM approximation of the functional — some nomenclature

- "Local" nodes, numbering schemes, coordinates etc. apply to a prototype (or master) element considered in isolation, frequently on local coordinate system of unit length.
- "Global" quantities refer to *connected* system, taking into account the actual geometry (element length here).
- Nodes and elements are numbered; some scheme is required to keep track of relationship between global elements, associated global node numbers and coordinates, and local nodes in each element.

FEM approximation of functional *contd.*

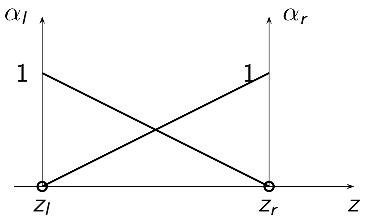
- Will use following local basis function of first order to approximate voltage on master element lying between z_l and z_r:

$$\tilde{V}^e = \alpha_l(z)V_l + \alpha_r(z)V_r \tag{7}$$

with

$$\alpha_l(z) = \frac{z_r - z}{z_r - z_l} \tag{8}$$

$$\alpha_r(z) = \frac{z - z_l}{z_r - z_l} \tag{9}$$



- Interpolation (basis) functions $\alpha_l(z)$ and $\alpha_l(z)$ illustrated here.
- V_l and V_r are voltages on *l*eft node and *r*ight node respectively — *and are unknowns to be determined by FEM, a.k.a. degrees of freedom.*

FEM approximation of functional *contd.*

- Substituting this linear approximation for V into energy functional, obtain for element e:

$$
\begin{aligned}
F^e(\tilde{V}^e) &= \frac{1}{2}\int_{z_l}^{z_r}\left[\frac{1}{L}(\frac{d\tilde{V}^e}{dz})^2 - \omega^2 C(\tilde{V}^e)^2\right] dz \\
&= \frac{1}{2}\int_{z_l}^{z_r}\left[\frac{1}{L}\left(\frac{d}{dz}\left[\alpha_l(z)V_l + \alpha_r(z)V_r\right]\right)^2 - \omega^2 C\left[\alpha_l(z)V_l + \alpha_r(z)V_r\right]^2\right] dz
\end{aligned}
\tag{10}
$$

- Can be written in matrix form as follows:

$$
F^e(\tilde{V}^e) = \begin{Bmatrix} V_l & V_r \end{Bmatrix} \left[\tfrac{1}{L_e}\mathcal{S} - \omega^2 C_e \mathcal{T}\right] \begin{Bmatrix} V_l \\ V_r \end{Bmatrix} \tag{11}
$$

Evaluating the elemental matrices

- Elemental matrices $\mathcal{S}$ and $\mathcal{T}$ appear so frequently in finite element analysis that they have commonly accepted names (but not symbols).
- From elasticity problems in structural mechanics, former is called *stiffness* matrix, and latter *mass* matrix.
- Increasingly used in CEM literature.
- Other names are *Dirichlet matrix* and *metric*, but these are not widely encountered.
- The elements of these matrices are given as follows:

$$\mathcal{S}_{ij} = \int_{z_l}^{z_r} \frac{d\alpha_i}{dz} \frac{d\alpha_j}{dz} \, dz \tag{12}$$

$$\mathcal{T}_{ij} = \int_{z_l}^{z_r} \alpha_i \alpha_j \, dz \tag{13}$$

with indices i and j taking both values l and r.

Evaluating the elemental matrices *contd.*

- Now introduce normalized local coordinates ξ_r and ξ_l:

$$\xi_r \equiv \frac{z - z_l}{z_r - z_l} = \frac{z - z_l}{h_e} = \alpha_r(z) \tag{14}$$

$$\xi_l = 1 - \xi_r = \alpha_l(z) \tag{15}$$

- This permits above to be written in general form, independent of specific global geometry — very widely used in FEA.
- Note that $\xi_r + \xi_r = 1$, i.e. *dependent* coordinates.
- In 1D, need only retain one; here, we choose $\xi = \xi_r$.
- Eqs. (12) & (13) are written i.t.o. global coordinates (z); re-write these normalized form.

Evaluating the elemental matrices *contd.*

- From $\frac{d\xi}{dz} = \frac{1}{h_e}$, integration variable becomes $dz = h_e d\xi$, and Eqs. (12) and (13) become:

$$\mathcal{S}_{ij} = h_e \int_0^1 \left(\frac{d\alpha_i}{d\xi} \frac{d\xi}{dz} \right) \left(\frac{d\alpha_j}{d\xi} \frac{d\xi}{dz} \right) d\xi$$

$$= \frac{1}{h_e} \int_0^h \frac{d\alpha_i}{d\xi} \frac{d\alpha_j}{d\xi} d\xi \qquad (16)$$

$$\mathcal{T}_{ij} = h_e \int_0^1 \alpha_i \alpha_j dz d\xi \qquad (17)$$

- Introducing normalized matrices $[S]$ and $[T]$, we can write

$$[\mathcal{S}] = \frac{1}{h_e}[S] \qquad (18)$$

$$[\mathcal{T}] = h_e[T] \qquad (19)$$

with entries

$$S_{ij} = \int_0^1 \frac{d\alpha_i}{d\xi} \frac{d\alpha_j}{d\xi} \; d\xi \qquad (20)$$

$$T_{ij} = \int_0^1 \alpha_i \alpha_j \; d\xi \qquad (21)$$

Evaluating the elemental matrices *contd.*

- For this linear element, the entries are very simple to evaluate. We will consider the evaluation of S_{lr} and T_{lr} in detail:

$$\begin{aligned} S_{lr} &= \int_0^1 \frac{d\alpha_l}{d\xi}\frac{d\alpha_r}{d\xi}\; d\xi \\ &= \int_0^1 (-1)(1)\; d\xi \\ &= -1 \end{aligned} \tag{22}$$

$$\begin{aligned} T_{lr} &= \int_0^1 \alpha_l \alpha_r\; d\xi \\ &= \int_0^1 (1-\xi)(\xi)\; d\xi \\ &= \left[\frac{1}{2}\xi^2 - \frac{1}{3}\xi^3\right]\Big|_0^1 \\ &= \frac{1}{6} \end{aligned} \tag{23}$$

Evaluating the elemental matrices *contd.*

- It is easily shown that the matrix is symmetric, and also that the diagonal entries are the same. In short, the normalized matrices have the following form:

$$[S] = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{24}$$

$$[T] = \frac{1}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{25}$$

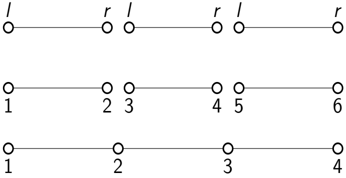
- This simple, closed form evaluation of the mass and stiffness matrices on a prototype element is typical of FEM — extends to 2D and 3D.
- Reason is absence of (singular) Green's function, so integrands are simple polynomials.
- One caveat is that with curvilinear (curved) elements, this is no longer possible; each element may be different.

9.2.5 Assembling the system

- Eq. (11) represents discretized energy-related functional in each element.
- Now, the elements must be brought together — "assembly-by-elements".
- Illustrate the approach using a *connection matrix* $[\mathcal{C}]$ to connect disconnected nodes to the global (connected) nodes:

$$\{V\}_{\mathrm{dis}} = [\mathcal{C}]\,\{V\}_{\mathrm{con}} \qquad (26)$$

A three-element example:



$$\begin{Bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{Bmatrix}_{\mathrm{dis}} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & & \\ & & 1 & \\ & & 1 & \\ & & & 1 \end{bmatrix}_{\mathrm{con}} \begin{Bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{Bmatrix}_{\mathrm{con}} \qquad (27)$$

Assembling the system *contd.*

- The functional for the whole ensemble of elements may now be written as a quadratic form:

$$F(\tilde{V}) = \{V\}_{\text{dis}}^{T} \begin{bmatrix} \frac{S}{h_1 L_1} - \omega^2 h_1 C_1 T & & & \\ & \frac{S}{h_2 L_2} - \omega^2 h_2 C_2 T & & \\ & & \ddots & \\ & & & \frac{S}{h_N L_N} - \omega^2 h_2 C_2 C_N T \end{bmatrix} \{V\}_{\text{dis}} \tag{28}$$

where T is the transpose operation.

- Can be written more compactly as

$$F(\tilde{V}) = \{V\}_{\text{dis}}^{T} [\mathcal{M}]_{\text{dis}} \{V\}_{\text{dis}} \tag{29}$$

Assembling the system *contd.*

- Write now i.t.o. connected voltage matrix as:

$$F(\tilde{V}) = \{V\}_{\mathrm{con}}^{T} [\mathcal{C}]^{T} [\mathcal{M}]_{\mathrm{dis}} [\mathcal{C}] \{V\}_{\mathrm{con}} \tag{30}$$

- Implies that matrix $[\mathcal{M}]$

$$[\mathcal{M}] = [\mathcal{C}]^{t} [\mathcal{M}]_{\mathrm{dis}} [\mathcal{C}] \tag{31}$$

can be viewed as coefficient matrix associated with connected problem, which is

$$F(\tilde{V}) = \{V\}_{\mathrm{con}}^{T} [\mathcal{M}] \{V\}_{\mathrm{con}} . \tag{32}$$

Assembling the system *contd.*

- If line is uniform ($L_e = L,\ C_e = C,\ h_e = h$), a simple result is obtained:

$$[\mathcal{M}] = \frac{1}{L}[\mathcal{C}]^t[\mathcal{S}][\mathcal{C}] - \omega^2 C[\mathcal{C}]^t[\mathcal{T}][\mathcal{C}] \tag{33}$$

- The constitutive parts of this matrix are given by

$$\frac{1}{L}[\mathcal{C}]^t[\mathcal{S}][\mathcal{C}] = \frac{1}{hL}\begin{bmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & & \ddots & & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix} \tag{34}$$

and

$$C[\mathcal{C}]^t[\mathcal{T}][\mathcal{C}] = \frac{hC}{6}\begin{bmatrix} 2 & 1 & & & & \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & \\ & & & \ddots & & \\ & & & 1 & 4 & 1 \\ & & & & 1 & 2 \end{bmatrix} \tag{35}$$

Rendering the functional stationary and solving the problem

- Final step is to make the functional stationary.
- Requires differentiating with respect to each degree of freedom.
- Now, finally, BCs on original domain must be taken into account.
- The RHS node (V_N) must be set to source voltage V_0.
- LHS node (homogeneous Neumann), turns out to be a *natural* boundary condition — which means that it will be satisfied automatically.
- Thus: differentiate Eq. (32) with respect to $V_1, V_2, \ldots, V_{N-1}$.
- For now, expand Eq. (32) & carry out differentiation explicitly.

Rendering functional stationary *contd.*

- Expanding the quadratic form in Eq. (32) results in

$$
\begin{aligned}
F = & M_{11} V_1^2 + M_{12} V_1 V_2 \\
& + M_{21} V_2 V_1 + M_{22} V_2^2 + M_{23} V_2 V_3 + \\
& + M_{32} V_3 V_2 + M_{33} V_3^2 + M_{34} V_3 V_4 + \\
& \vdots \\
& \ldots + M_{N,N-1} V_{N-1} V_N \\
& + M_{N-1,N} V_N V_{N-1} + M_{N,N} V_N^2
\end{aligned}
\tag{36}
$$

- Differentiate with respect to $V_1, V_2, \ldots, V_{N-1}$ in turn — but *not* w.r.t. V_N, which is prescribed – and set each equal to zero.

Rendering functional stationary *contd.*

- Using symmetry, $M_{ij} = M_{ji}$, collecting result into a matrix equation & simplifying, one obtains following $N-1 \times N-1$ system:

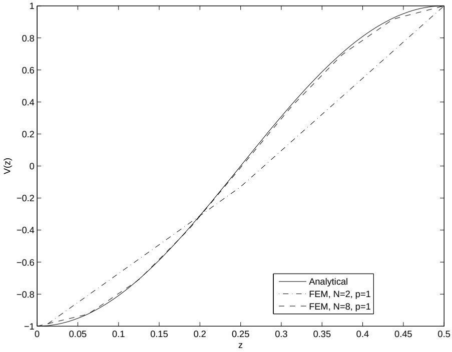
$$\frac{\partial F}{\partial\{V_1, V_2, \dots, V_{N-1}\}}^T = \begin{bmatrix} M_{11} & M_{12} & & \\ M_{21} & M_{22} & M_{23} & \\ & \ddots & & \\ & & & M_{N-1,N-1} \end{bmatrix} \begin{Bmatrix} V_1 \\ V_2 \\ \vdots \\ V_{N-1} \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ -M_{N-1,N} V_N \end{Bmatrix} \tag{37}$$

- As with MoM, this is a *linear system of equations* which can be solved using standard tools.
- *Unlike* the MoM, the FEM system matrix is (highly) sparse; efficient matrix solvers exploit this. (Note that in this particular case, the system matrix is tri-diagonal, but this is not a general result).

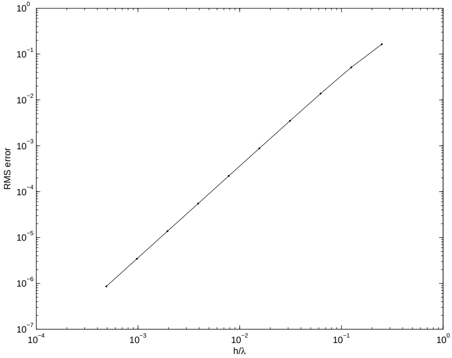
9.2.8 Results

- Results computed with code shown opposite.
- $L = C = 1$, so $Z_0 = 1\Omega$, $v_p = 1\text{m/s}$. $f = 1\text{Hz}$, thus $\lambda = 1\text{m}$.
- Figure shows results computed with 2 and 8 elements. Former gives poor approximation of the problem; the natural BC on the LHS is very poorly satisfied.
- However, with as few as 8 elements, a relatively good solution has already been obtained.



9.2.8 Rate of convergence

- Quantitative assessment of error is required; one's eye quickly cannot distinguish errors as mesh is refined.
- Shown here is RMS of error norm (computed from linear interpolation of FEM solution w.r.t. analytical solution).
- From theoretical considerations, the expected rate of convergence (for smooth solution) is $\mathcal{O}(h^2)$.
- Intuitively expected, as linear approximation is complete to first order, so error terms expected to be of 2nd order.



- First-order polynomial fit to the log-log graph gives a slope of $1.97 \approx 2$.

Conclusions

- Simple time-harmonic problem has illustrated basic concepts of FEA.
- Results are quite satisfactory, but there are a number of limitations which can be addressed.
- Two obvious ones are:
  - Extension to use *higher-order* finite elements.
  - Incorporation of more general boundary conditions.
- Both can be readily addressed.
- Reference: DB Davidson, *Computational Electromagnetics for RF and Microwave Engineering*, Cambridge University Press, 2nd edition, 2011, Chapter 9.